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Shoichiro Sakai
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Preliminaries

In this chapter we shall state some fundamental facts used in this book. We shall also use elementary facts on C*-algebra and W*-algebras freely. The reader can find the proofs of them in the author's book [165] or other standard text books [41], [42], [139], [193], [194], [195], [196]. In addition we shall use some common notation without definition. The reader can find the appropriate definitions in [165].

1.1 Banach algebras, C*-algebras and W*-algebras

Let $\mathscr{A}$ be a linear associative algebra over the complex numbers. The algebra $\mathscr{A}$ is called a normed algebra if there is associated to each element x a real number $\|x\|$, called the norm of x, with the properties:

(1) $\|x\| \geqslant 0$, and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|x + y\| \leqslant \|x\| + \|y\|$;
(3) $\|\lambda x\| = |\lambda| \|x\|$, where λ is a complex number;
(4) $\|xy\| \leqslant \|x\| \|y\|$.

If $\mathscr{A}$ is complete with respect to the norm (i.e., if $\mathscr{A}$ is also a Banach space), then it is called a Banach algebra. A mapping $x \mapsto x^*$ of $\mathscr{A}$ onto $\mathscr{A}$ itself is called an involution if it satisfies the following conditions:

(1) $(x^*)^* = x$;
(2) $(x + y)^* = x^* + y^*$;
(3) $(xy)^* = y^* x^*$;
(4) $(\lambda x)^* = \bar{\lambda} x^*$, λ a complex number.

An algebra with an involution is called a *-algebra. A Banach *-algebra is called a C*-algebra if it satisfies $\|x^* x\| = \|x\|^2$ for $x \in \mathscr{A}$. It is easily seen that $\|x^* x\| = \|x\|^2$ for $x \in \mathscr{A}$ implies $\|x^*\| = \|x\|$ $(x \in \mathscr{A})$.

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A C*-algebra $\mathscr{M}$ is called a W*-algebra if it is a dual space as a Banach space (i.e. there is a Banach space $\mathscr{M}_*$ such that the dual $(\mathscr{M}_*)^*$ of $\mathscr{M}_*$ is $\mathscr{M}$). We call $\mathscr{M}_*$ the *pre-dual* of $\mathscr{M}$. In general, a dual Banach space is not necessarily the dual space of a unique Banach space. However the W*-algebra has a unique pre-dual space. The W*-algebras are also called von Neumann algebras and the present definition (W*-algebras) agrees with the familiar alternative definitions of von Neumann algebras. This Theorem is due to the author.

1.2 Topologies on C*-algebras and W*-algebras

The topology defined by the norm $\|\cdot\|$ on a C*-algebra $\mathscr{A}$ is called the uniform topology (or the norm topology). The weak *-topology $\sigma(\mathscr{M}, \mathscr{M}_*)$ on $\mathscr{M}$ is called the weak topology or the σ-weak topology.

1.3 *-Homomorphisms, *-isomorphisms and *-automorphisms

Let $\mathscr{A}$, $\mathscr{B}$ be C*-algebras. A linear mapping Φ of $\mathscr{A}$ into $\mathscr{B}$ is said to be a *-homomorphism if it satisfies:

(1) $\Phi(xy) = \Phi(x)\Phi(y)$;
(2) $\Phi(x^*) = \Phi(x)^*$.

A *-homomorphism Φ of a C*-algebra $\mathscr{A}$ into $\mathscr{B}$ is always norm-decreasing (i.e. $\|\Phi(x)\| \leqslant \|x\|$). Moreover the image $\Phi(\mathscr{A})$ is automatically uniformly closed in $\mathscr{B}$.

A one-to-one *-homomorphism of $\mathscr{A}$ into $\mathscr{B}$ is called a *-isomorphism. A *-isomorphism Φ is always isometric (i.e. $\|\Phi(x)\| = \|x\|$).

A *-isomorphism ρ of $\mathscr{A}$ onto $\mathscr{A}$ itself is called a *-automorphism of $\mathscr{A}$. Let $\mathscr{M}$, $\mathscr{N}$ be W*-algebras and let Φ be a *-isomorphism of $\mathscr{M}$ onto $\mathscr{N}$; then Φ is bicontinuous with respect to $\sigma(\mathscr{M}, \mathscr{M}_*)$ and $\sigma(\mathscr{N}, \mathscr{N}_*)$.

1.4 Self-adjointness and positivity

An element a in a C*-algebra $\mathscr{A}$ is said to be self-adjoint if $a^* = a$. An element h in a C*-algebra $\mathscr{A}$ is said to be positive if it can be expressed as $h = a^*a$ for some $a \in \mathscr{A}$. It can be shown that the set of all positive elements in $\mathscr{A}$ forms a convex cone.

1.5 Positive linear functionals and states

A linear functional f on a C*-algebra $\mathscr{A}$ is said to be positive if $f(x^*x) \geqslant 0$ for $x \in \mathscr{A}$. The reader can readily verify the following facts. If $\mathscr{A}$ has an

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identity, then a positive linear functional is bounded and $\|f\| = f(1)$. A bounded linear functional f on $\mathscr{A}$ is positive if there exists a positive element h (>0) in $\mathscr{A}$ such that $f(h) = \|f\|\,\|h\|$.

A positive linear functional ϕ on $\mathscr{A}$ is said to be a *state* if $\|\phi\| = 1$.

A state ϕ on $\mathscr{A}$ is said to be *tracial* if $\phi(xy) = \phi(yx)$ for $x, y \in \mathscr{A}$.

1.6 Commutative C*- and W*-algebras

This section gives the spectral representation for commutative algebras (both C* and W*). Let Ω be a locally compact Hausdorff space, and let $C_0(\Omega)$ be the algebra of all complex-valued continuous functions on Ω vanishing at infinity. Define $\|a\| = \sup_{t\in\Omega}|a(t)|$ and $a^*(t) = \overline{a(t)}$ for $a \in C_0(\Omega)$. Then $C_0(\Omega)$ is a commutative C*-algebra. If K is a compact Hausdorff space, $C(K)$ (the algebra of all complex-valued continuous functions on K) is a commutative C*-algebra with identity. The converse statement is also true: If $\mathscr{A}$ is a given commutative C*-algebra, then it is *-isomorphic to $C_0(\Omega)$, where Ω is some locally compact Hausdorff space; if $\mathscr{A}$ is assumed to be commutative with identity, then $\mathscr{A}$ is *-isomorphic to $C(K)$, where K is a compact Hausdorff space.

Let (Ω, μ) be a given measure space with $\mu(\Omega) < +\infty$, let $L^\infty(\Omega, \mu)$ be the C*-algebra of all essentially bounded μ-measurable functions on Ω and let $L^1(\Omega, \mu)$ be the Banach space of all μ-integrable functions on Ω. Then by the Random–Nikodym theorem, $L^1(\Omega, \mu)^* = L^\infty(\Omega, \mu)$. Hence $L^\infty(\Omega, \mu)$ is a commutative W*-algebra. More generally, let (Γ, ν) be a localizable measure space (i.e. a direct sum of finite measure spaces) and let $L^\infty(\Gamma, \nu)$ be the C*-algebra of all ν-essentially bounded measurable functions on Γ, and let $L^1(\Gamma, \nu)$ be the Banach space of all ν-integrable functions on Γ. Then $(L^1(\Gamma, \nu))^* = L^\infty(\Gamma, \nu)$; hence $L^\infty(\Gamma, \nu)$ is a commutative W*-algebra.

Conversely, if $\mathscr{M}$ is a commutative W*-algebra, then it is *-isomorphic to $L^\infty(\Gamma, \nu)$, where (Γ, ν) is a localizable measure space. Hence we also have a spectral representation for the commutative W*-algebras.

1.7 Concrete C*- and W*-algebras

Let $\mathscr{H}$ be a complex Hilbert space and let $B(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. We can define various topologies on $B(\mathscr{H})$. Although well known, they will be reviewed below.

(1) The uniform topology This is given by the operator norm $\|a\|$ ($a \in B(\mathscr{H})$), where $\|a\| = \sup_{\substack{\|\xi\|\leq 1\\ \xi\in\mathscr{H}}} \|a\xi\|$. Take the adjoint operation $a \mapsto a^*$ as the involution* (i.e. $(a\xi, \eta) = (\xi, a^*\eta)$ $(\xi, \eta \in \mathscr{H})$), where (,) is the scalar product (inner product) of $\mathscr{H}$). Then, $B(\mathscr{H})$ is a C*-algebra and moreover *any*

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uniformly closed self-adjoint subalgebra $\mathscr{A}$ (i.e. $a \in \mathscr{A}$ implies $a^* \in \mathscr{A}$) of $B(\mathscr{H})$) is itself a C*-algebra.

(2) The strong operator topology Let $\xi \in \mathscr{H}$. The function $a \mapsto \|a\xi\|$ is then a semi-norm on $B(\mathscr{H})$. The set of all such semi-norms $\{\|a\xi\| \mid \xi \in \mathscr{H}\}$ defines a Hausdorff locally convex topology on $B(\mathscr{H})$, called the strong operator topology.

(3) The weak operator topology For each pair $\xi, \eta \in \mathscr{H}$, the function $a \to |(a\xi, \eta)|$ defines a semi-norm on $B(\mathscr{H})$. The set of all such semi-norms $\{|(a\xi, \eta)| \mid \xi, \eta \in \mathscr{H}\}$ defines a Hausdorff locally convex topology on $B(\mathscr{H})$, called the weak operator topology.

(4) The σ-weak topology Let Tr be the trace function in $B(\mathscr{H})$ and let $T(\mathscr{H})$ be the set of all trace class operators on $\mathscr{H}$. For $a \in T(\mathscr{H})$, define $\|a\|_1 = \mathrm{Tr}(|a|)$, where $|a| = (a^*a)^{1/2}$. Then $T(\mathscr{H})$ becomes a Banach space with the norm $\|\cdot\|_1$. Let f be a bounded linear functional on $B(\mathscr{H})$ which is assumed to be continuous on bounded spheres of $B(\mathscr{H})$ with respect to the weak operator topology. Then there is a unique trace class operator a on $\mathscr{H}$ such that $f(x) = \mathrm{Tr}(xa)$ for $x \in B(\mathscr{H})$ and $\|f\| = \mathrm{Tr}(|a|)$. Therefore $T(\mathscr{H})$ can be identified with the Banach space of all bounded linear functionals of $B(\mathscr{H})$ which are continuous with respect to the weak operator topology on bounded spheres of $B(\mathscr{H})$. Then it is known that $T(\mathscr{H})^* = B(\mathscr{H})$. Hence $B(\mathscr{H})$ is a W*-algebra and $\sigma(B(\mathscr{H}), B(\mathscr{H})_*) = \sigma(B(\mathscr{H}), T(\mathscr{H}))$ is the σ-weak topology of $B(\mathscr{H})$.

Let $\mathscr{M}$ be a self-adjoint subalgebra of $B(\mathscr{H})$ which is closed with respect to the weak operator topology; then it is $\sigma(B(\mathscr{H}), B(\mathscr{H})_*)$-closed. Let $\mathscr{M}^0$ be the polar of $\mathscr{M}$ in $T(\mathscr{H})$; then $(T(\mathscr{H})/\mathscr{M}^0)^* = \mathscr{M}$. Hence $\mathscr{M}$ is a W*-algebra and the restriction of $\sigma(B(\mathscr{H}), B(\mathscr{H})_*)$ to $\mathscr{M}$ is the σ-weak topology on $\mathscr{M}$.

Let $\mathscr{M}$ be a self-adjoint subalgebra of $B(\mathscr{H})$; then the following six properties are equivalent:

(1) $\mathscr{M} \cap S$ is closed with respect to the weak operator topology, where S is the unit sphere of $B(\mathscr{H})$;
(2) $\mathscr{M}$ is closed with respect to the weak operator topology;
(3) $\mathscr{M} \cap S$ is closed with respect to the strong operator topology;
(4) $\mathscr{M}$ is closed with respect to the strong operator topology;
(5) $\mathscr{M} \cap S$ is $\sigma(B(\mathscr{H}), B(\mathscr{H})_*)$-closed;
(6) $\mathscr{M}$ is $\sigma(B(\mathscr{H}), B(\mathscr{H})_*)$-closed.

The weak operator topology and $\sigma(\mathscr{M}, \mathscr{M}_*)$ are equivalent on $\mathscr{M} \cap S$. Therefore without confusion, we can call a self-adjoint subalgebra of $B(\mathscr{H})$, which is closed with respect to the weak operator topology or $\sigma(B(\mathscr{H}), B(\mathscr{H})_*)$, a weakly closed self-adjoint subalgebra of $B(\mathscr{H})$.

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1.8 Representation theorems for C*- and W*-algebras

(1) Let $\mathscr{A}$ be a C*-algebra; then $\mathscr{A}$ is *-isomorphic to a uniformly closed self-adjoint subalgebra of $B(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. (the Gelfand–Naimark theorem).

(2) Let $\mathscr{M}$ be a W*-algebra; then $\mathscr{M}$ is *-isomorphic to a weakly closed self-adjoint subalgebra of $B(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. Moreover $\sigma(\mathscr{M}, \mathscr{M}_*)$ coincides with the restriction of $\sigma(B(\mathscr{H}), B(\mathscr{H})_*)$ under the *-isomorphism.

The next two theorems are fundamental to operator algebra theory. They will be stated without proof.

1.9 Commutation theorem (von Neumann's double commutant theorem)

Let $\mathscr{M}$ be a self-adjoint subalgebra of $B(\mathscr{H})$ containing the identity operator on a Hilbert space $\mathscr{H}$. Then, $\mathscr{M}$ is weakly closed if and only if $\mathscr{M}'' = \mathscr{M}$, where $\mathscr{M}' = \{x \in B(\mathscr{H}) | ax = xa \text{ for } a \in \mathscr{M}\}$ and $\mathscr{M}'' = (\mathscr{M}')'$.

1.10 Kaplansky's density theorem (Bounded approximation)

Let $\mathscr{B}$ be a self-adjoint subalgebra of $B(\mathscr{H})$ and let $\bar{\mathscr{B}}$ be the closure of $\mathscr{B}$ in $B(\mathscr{H})$ with respect to the weak operator topology; then $\bar{\mathscr{B}}$ is a weakly closed self-adjoint subalgebra of $B(\mathscr{H})$ and moreover $\bar{\mathscr{B}} \cap S$ is contained in the closure of $\mathscr{B} \cap S$ with respect to the strong operator topology, where S is the unit sphere of $B(\mathscr{H})$.

1.11 GNS (Gelfand–Naimark–Segal) representations

(1) From state to representation Let ϕ be a bounded positive linear functional on a C*-algebra $\mathscr{A}$. Introduce a conjugate bilinear functional $(x, y) = \phi(y^*x)$ in $\mathscr{A}$. Let $\mathscr{I} = \{x | \phi(x^*x) = 0, x \in \mathscr{A}\}$. Then $\mathscr{I}$ is a closed left ideal of $\mathscr{A}$. Define a conjugate bilinear functional on the quotient linear space $\mathscr{A}/\mathscr{I}$ such that if $x \in x_\phi$, $y \in y_\phi$, then $(x_\phi, y_\phi) = \phi(y^*x)$ (here x_ϕ(resp. y_ϕ) is the class containing x(resp. y)). The expression (x_ϕ, y_ϕ) does not then depend on a special choice of the representatives x, y. It will define a scalar product on $\mathscr{A}/\mathscr{I}$ under which $\mathscr{A}/\mathscr{I}$ will become a pre-Hilbert space. Let $\mathscr{H}_\phi$ be the completion with respect to this scalar product. Then $\mathscr{H}_\phi$ is a Hilbert space. Now we shall construct a *-representation of $\mathscr{A}$ on $\mathscr{H}_\phi$ (i.e. a *-homomorphism of $\mathscr{A}$ into $B(\mathscr{H}_\phi)$) via ϕ, denoted by $\{\pi_\phi, \mathscr{H}_\phi\}$. Put $\pi_\phi(a)x_\phi = (ax)_\phi$. Then $\pi_\phi(a)$ $(a \in \mathscr{A})$ is a linear operator on $\mathscr{A}/\mathscr{I}$. Moreover it is a bounded linear operator on the pre-Hilbert space $\mathscr{A}/\mathscr{I}$;

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hence it can be uniquely extended to a bounded linear operator on $\mathscr{H}_\phi$, denoted by $\pi_\phi(a)$ again. Then, $a \mapsto \pi_\phi(a)$ $(a \in \mathscr{A})$ is a *-homomorphism of $\mathscr{A}$ into $B(\mathscr{H}_\phi)$. The *-representation $\{\pi_\phi, \mathscr{H}_\phi\}$ is called the GNS representation constructed via ϕ. It is known that there is a vector ξ_0 in $\mathscr{H}_\phi$ such that $\phi(x) = (x\xi_0, \xi_0)$ for $x \in \mathscr{A}$, and moreover $[\pi_\phi(\mathscr{A})\xi_0] = \mathscr{H}_\phi$ (i.e. $\{\pi_\phi, \mathscr{H}_\phi\}$ is cyclic), where $[(\cdot)]$ is the closure of $(\cdot)$ in $\mathscr{H}_\phi$.

If $\mathscr{A}$ has an identity 1 and ϕ is a state on $\mathscr{A}$, then $\xi_0 = 1_\phi$.

(2) From representation to state Conversely, if π is a *-representation of $\mathscr{A}$ into $B(\mathscr{H})$ such that $[\pi(\mathscr{A})\xi_0] = \mathscr{H}$, then $\{\pi, \mathscr{H}\}$ is unitarily equivalent to $\{\pi_\phi, \mathscr{H}_\phi\}$, where $\varphi(x) = (x\xi_0, \xi_0)$ $(x \in \mathscr{A})$.

(3) Normality Let ψ be a positive linear functional on a W*-algebra $\mathscr{M}$. ψ is then said to be *normal* if it is $\sigma(\mathscr{M}, \mathscr{M}_*)$-continuous. Let ψ be a normal positive linear functional on $\mathscr{M}$; then the GNS representation $\{\pi_\psi, \mathscr{H}_\psi\}$ satisfies the following conditions:

(1) $\pi_\psi(\mathscr{M})$ is weakly closed in $B(\mathscr{H}_\psi)$;
(2) π_ψ is a continuous mapping of $\mathscr{M}$ with $\sigma(\mathscr{M}, \mathscr{M}_*)$ into $B(\mathscr{H}_\psi)$ with $\sigma(B(\mathscr{H}_\psi), B(\mathscr{H}_\psi)_*)$.

1.12 Factorial and pure states

A W*-algebra $\mathscr{M}$ is called a factor if its center consists of scalar multiples of the identity. Let $\mathscr{A}$ be a C*-algebra and let ϕ be a state on $\mathscr{A}$. ϕ is said to be *factorial if* $\pi_\phi(\mathscr{A})''$ is a factor; ϕ is said to be pure if $\pi_\phi(\mathscr{A})'' = B(\mathscr{H}_\phi)$.

Let $\mathscr{S}_\mathscr{A}$ be the set of all states on $\mathscr{A}$; then a state ϕ on $\mathscr{A}$ is pure if and only if ϕ is an extreme point in the convex set $\mathscr{S}_\mathscr{A}$. If $\mathscr{A}$ has an identity, then $\mathscr{S}_\mathscr{A}$ is $\sigma(\mathscr{A}^*, \mathscr{A})$-compact. The space $\mathscr{S}_\mathscr{A}$ with $\sigma(\mathscr{A}^*, \mathscr{A})$ is called the state space of $\mathscr{A}$.

In Chapter 4, we shall often use harmonic functions, and so we shall now state the fundamental facts on harmonic function, which we shall need in Chapter 4. Let $\mathbb{C}$ be the complex plane, and let D be the closed unit disk in $\mathbb{C}$ such that $D = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. Let β be a non-zero real number and let S_β be the closed strip of the complex plane such that

$$S_\beta = \{z \in \mathbb{C} \mid 0 \leqslant \operatorname{Im}(z) \leqslant \beta \text{ if } \beta > 0 \text{ or } \beta \leqslant \operatorname{Im}(z) \leqslant 0 \text{ if } \beta < 0\}.$$

Let D^0 (resp. ${S_\beta}^0$) be the interior of D (resp. S_β) and let $D_1 = \{z \in D \mid z \neq -1, 1\}$. Consider a mapping $\omega = (\beta/\pi)\log \mathrm{i}(1-z)/(1+z)$ of D_1 in the z-complex plane onto S_β in the ω-complex plane. Then it is a one-to-one bicontinuous mapping of D_1 onto S_β and moreover it is a conformal mapping of D^0 onto ${S_\beta}^0$. Therefore, by Poisson's formula and the result from the Dirichlet problem, we have the following theorem (cf. [77]).

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1.13 Theorem (Poisson Kernel for the strip)

There exist two positive continuous functions $K_1(z,t), K_2(z,t)$ *on* $S_\beta{}^0 \times \mathbb{R}$ *such that*

$$\int_{-\infty}^{\infty} K_1(z,t)\,\mathrm{d}t + \int_{-\infty}^{\infty} K_2(z,t)\,\mathrm{d}t = 1 \qquad \text{for } z \in S_\beta{}^0$$

and for any two real-valued, bounded continuous functions f_1, f_2 *on* $\mathbb{R}$,

$$u(z) = \int_{-\infty}^{\infty} f_1(t) K_1(z,t)\,\mathrm{d}t + \int_{-\infty}^{\infty} f_2(t) K_2(z,t)\,\mathrm{d}t \qquad \text{for } z \in S_\beta{}^0$$

is harmonic on $S_\beta{}^0$ *and moreover*

$$\lim_{\substack{z \to t \\ z \in S_\beta{}^0}} u(z) = f_1(t) \qquad \textit{and} \qquad \lim_{\substack{z \to t + i\beta \\ z \in S_\beta{}^0}} u(z) = f_2(t) \, (t \in \mathbb{R}).$$

Furthermore if u_1 *is a bounded continuous function on* S_β *which is harmonic on* $S_\beta{}^0$, *and* $u_1(t) = f_1(t)$, $u_1(t + i\beta) = f_2(t)$ $(t \in R)$, *then* $u_1(z) = u(z)$ *for* $z \in S_\beta$.

We shall not use the explicit form for the functions K_1, and K_2 but only the representation theorems stated here and their corollary:

1.14 Corollary (analytic version)

Let $f(z)$ *be a bounded continuous function on* S_β *which is analytic on* $S_\beta{}^0$; *then*

$$f(z) = \int_{-\infty}^{\infty} f(t) K_1(z,t)\,\mathrm{d}t + \int_{-\infty}^{\infty} f(t + \mathrm{i}\beta) K_2(z,t)\,\mathrm{d}t \qquad \textit{for } z \in S_\beta{}^0,$$

Perturbation Theory

We shall also need some fundamental results on bounded perturbations in Chapter 4. Here we mention some of them.

Let A, B be bounded linear operators on a Banach space E. Let $u_t = \exp(tA)$ and $T_t = \exp t(A+B)(t \in R)$ be exponential one-parameter groups.

1.15 Perturbation expansion theorem ($t \in \mathbb{R}$)

We have

$$T_t = u_t + \sum_{n=1}^{\infty} \int_{0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t} u_{t_1} B u_{t_2 - t_1} B u_{t_3 - t_2} \cdots u_{t_n - t_{n-1}} B u_{t - t_n} \,\mathrm{d}t_1\,\mathrm{d}t_2 \cdots \mathrm{d}t_n.$$

where the series is norm-convergent for all t.

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Proof Let $T_t^{(0)} = u_t$ and

$$T_t^{(n)} = \int_{0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t} u_{t_1} B u_{t_2 - t_1} B_{t_3 - t_2} \cdots u_{t_n - t_{n-1}} B u_{t - t_n} \, dt_1 \, dt_2 \cdots dt_n.$$

Then

$$\begin{aligned} T_t^{(n)} &= \int_0^t u_{t_1} B \, dt_1 \int_{t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t} u_{t_2 - t_1} B u_{t_3 - t_2} \cdots u_{t_n - t_{n-1}} B u_{t - t_n} \, dt_2 \, dt_3 \cdots dt_n \\ &= \int_0^t u_{t_1} B \, dt_1 \int_{0 \leqslant t_2 \leqslant t_3 \leqslant \cdots \leqslant t_n \leqslant t - t_1} u_{t_2} B_{t_3 - t_2} \cdots u_{t_n - t_{n-1}} B u_{t - t_1 - t_n} \, dt_2 \, dt_3 \cdots dt_n \\ &= \int_0^t u_{t_1} B T_{t - t_1}{}^{(n-1)} \, dt_1. \end{aligned}$$

On the other hand,

$$\left\| \sum_{n=0}^{\infty} T_t^{(n)} \right\| \leqslant \sum_{n=0}^{\infty} \left\| T_t^{(n)} \right\| \leqslant \sum_{n=0}^{\infty} \frac{M^{n+1} \| B \|^n}{n!} t^n,$$

where $M = \sup_{|s| \leqslant t} \| u_s \|$.

Therefore,

$$\sum_{n=1}^{\infty} T_t^{(n)} = \int_0^t u_{t_1} B \sum_{n=0}^{\infty} T_{t - t_1}{}^{(n)} \, dt_1.$$

Put $w_t = \sum_{n=0}^{\infty} T_t^{(n)}$; then by the previous equality, we have

$$w_t = u_t + \int_0^t u_s B w_{t-s} \, ds.$$

Hence,

$$\begin{aligned} w_{t_1} w_{t_2} &= \left(u_{t_1} + \int_0^{t_1} u_s B w_{t_1 - s} \, ds \right) w_{t_2} \\ &= u_{t_1} w_{t_2} + \int_0^{t_1} u_s B w_{t_1 - s} w_{t_2} \, ds \\ &= u_{t_1} \left(u_{t_2} + \int_0^{t_2} u_s B w_{t_2 - s} \, ds \right) + \int_0^{t_1} u_s B w_{t_1 - s} w_{t_2} \, ds \\ &= u_{t_1 + t_2} + \int_0^{t_2} u_{s + t_1} B w_{t_2 - s} \, ds + \int_0^{t_1} u_s B w_{t_1 - s} w_{t_2} \, ds \\ &= u_{t_1 + t_2} + \int_{t_1}^{t_1 + t_2} u_s B w_{t_1 + t_2 - s} \, ds + \int_0^{t_1} u_s B w_{t_1 - s} w_{t_2} \, ds \end{aligned}$$

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$$= u_{t_1+t_2} + \int_0^{t_1+t_2} u_s B w_{t_1+t_2-s}\,ds + \int_0^{t_1} u_s B(w_{t_1-s}w_{t_2} - w_{t_1+t_2-s})\,ds$$

$$= w_{t_1+t_2} + \int_0^{t_1} u_s B(w_{t_1-s}w_{t_2} - w_{t_1+t_2-s})\,ds.$$

If we replace B by $zB(z\in\mathbb{C})$, denote the corresponding w by w^z and put $G_{t_1}(z) = w_{t_1}{}^z w_{t_2}{}^z - w_{t_1+t_2}{}^z\ (z\in\mathbb{C})$, then $G_{t_1}(z)$ is entirely analytic with respect to z, and

$$G_{t_1}(z) = z\int_0^{t_1} u_s B G_{t_1-s}(z)\,ds = z\int_0^{t_1} u_{t_1-s} B G_s(z)\,ds.$$

Since $G_t(z) = \sum_{n=0}^{\infty}(G_t{}^{(n)}(0)/n!)z^n$,

$$\sum_{n=0}^{\infty}\frac{G_t{}^{(n)}(0)}{n!}z^n = z\int_0^t u_{t-s}B\sum_{n=0}^{\infty}\frac{G_s{}^{(n)}(0)}{n!}z^n\,ds$$

$$= \sum_{n=0}^{\infty} z^{n+1}\int_0^t u_{t-s}B\frac{G_s{}^{(n)}(0)}{n!}\,ds \qquad \text{for } z\in\mathbb{C}.$$

Hence $G_t{}^{(0)}(0) = G_t{}^{(0)}(0) = \cdots = G_t{}^{(n)}(0) = \cdots = 0$ and so $G_t(z) = 0$. Therefore, $w_{t_1+t_2} = w_{t_1}w_{t_2}$.

On the other hand,

$$w_t = u_t + \int_0^t u_s B w_{t-s}\,ds = u_t + \int_0^t u_{t-s} B w_s\,ds$$

$$= u_t + u_t\int_0^t u_{-s} B w_s\,ds.$$

Hence

$$\frac{d}{dt}w_t|_{t=0} = \frac{d}{dt}u_t|_{t=0} + u_t u_{-t} B w_t|_{t=0}$$

$$= A + B.$$

Therefore $w_t = \exp t(A+B)$. This completes the proof. □

1.16 Corollary (complex version)

We have

$$\exp z(A+B) = \exp(zA) + \sum_{n=1}^{\infty} z^n \int_{0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant 1} u_{t_1 z} B u_{(t_2-t_1)z} B\cdots$$

$$\times u_{(t_n-t_{n-1})z} B u_{(1-t_n)z}\,dt_1\,dt_2\cdots dt_n. \qquad (z\in\mathbb{C})$$

where the series is norm convergent for all z.

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Proof

$$\begin{aligned}\exp z(A+B) &= \exp 1(zA+zB)\\ &= \exp zA + \sum_{n=1}^{\infty}\int_{0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant 1} u_{t_1 z}(zB)u_{(t_2-t_1)z}(zB)u_{(t_3-t_2)z}\cdots\\ &\quad\times u_{(t_n-t_{n-1})z}(zB)u_{(1-t_{n-1})z}\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_n\\ &= \exp zA + \sum_{n=1}^{\infty} z^n\int_{0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant 1} u_{t_1 z}Bu_{(t_2-t_1)z}Bu_{(t_3-t_2)z}\cdots\\ &\quad\times u_{(t_n-t_{n-1})z}Bu_{(1-t_n)z}\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_n.\end{aligned}$$

This completes the proof. □

Next we shall consider a generalization of Theorem 1.15. Let A be a self-adjoint linear operator in a Hilbert space $\mathscr{H}$ and let B be a bounded self-adjoint linear operator on $\mathscr{H}$. The two operators A and B will be specified this way throughout the discussion to follow.

Geometric vectors

A vector ξ in $\mathscr{H}$ is said to be geometric with respect to A if there is a positive number M_ξ such that $\|A^n\xi\| \leqslant M_\xi{}^n\|\xi\|$ for $n = 1, 2, 3, \ldots$. Let $G(A)$ be the set of all geometric elements in $\mathscr{H}$ with respect to A. Let $A = \int_{-\infty}^{\infty}\lambda\,\mathrm{d}E_\lambda$ be the spectral decomposition of A and let $E_n = \int_{-n}^{n}\lambda\,\mathrm{d}E_\lambda$; then $\bigcup_{n=1}^{\infty}E_n\mathscr{H}\subset G(A)$ and so $G(A)$ is dense in $\mathscr{H}$.

Now assume that (1) $B\cdot G(A)\subset G(A)$; then for $\xi\in G(A)$,

$$(\exp t_1A)B(\exp(t_2-t_1)A)B\cdots(\exp(t_n-t_{n-1})A)B(\exp(t-t_n)A)\xi$$

is well-defined and it belongs to $G(A)$. In fact, for $t\in\mathbb{R}$, $\exp(tA)\xi = \sum_{n=0}^{\infty}(t^n/n!)A^n\xi$, and $\sum_{n=0}^{\infty}\|(t^n/n!)A^n\xi\| \leqslant \sum_{n=0}^{\infty}(M_\xi{}^n|t|^n/n!)\|\xi\| = \exp(M_\xi|t|)\|\xi\|$; hence $\exp(tA)\xi\in\mathscr{H}$. Moreover,

$$A^m\exp(tA)\xi = \sum_{n=0}^{\infty}\frac{t^n}{n!}A^{m+n}\xi$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{|t|^n}{n!}\|A^{m+n}\xi\| &\leqslant M_\xi{}^m\sum_{n=0}^{\infty}\frac{M_\xi{}^n|t|^n}{n!}\|\xi\|\\ &= M_\xi{}^m\exp(M_\xi|t|)\|\xi\|.\end{aligned}$$

Now put $\|\xi\| = V_t\|(\exp tA)\xi\|$; then $\|A^m\exp(tA)\xi\| \leqslant M_\xi{}^m\exp(M_\xi|t|)$

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