PART 1

FOUR-MANIFOLDS AND ALGEBRAIC SURFACES

The last few years have seen important advances in our understanding of 4-manifolds: their topology, differential topology and geometry. On the topological side there is a good picture of the full classification, through Freedman's h-cobordism and restricted s-cobordism theorems. In the differential topological category we are now well-aquainted with the special features of 4-manifold theory which are detected by the instanton solutions of the Yang-Mills equations, but the general classification is, for the moment, a matter of speculation. The 4-manifolds underlying complex algebraic surfaces have always provided a particularly interesting stock of examples, and the fascinating problems of understanding the interaction between the complex structure and the differential topology lie at the forefront of current research. One can obtain a good idea of the present position of the subject, and of the progress that has been made in recent years, by reading the two survey articles [M], [FM].

The five articles in this section cover many facets of the subject. The paper of Donaldson contains a general account of the use of Yang-Mills moduli spaces to define 4-manifold invariants, and some discussion of geometrical apects of the theory. In particular it gives a brief summary of the link between Yang-Mills theory over complex surfaces and stable holomorphic bundles, which in large measure accounts for the prominence of algebraic surfaces in the results. The paper of Gompf surveys the general picture of smooth 4-manifolds, especially algebraic surfaces, and presents partial classification results. It also contains wonderfully explicit "Kirby calculus" descriptions of some distinct differentiable structures on a family of open 4-manifolds, and ties these in to the ideas of Floer homology which we consider at greater length in the next section. The paper of Kotschick takes a more algebro-geometric stance, and surveys what is known about the differential topology of a special, but very important, class of complex surfaces. This class includes the "Dolgachev surfaces", which provided some of the first applications of the new techniques from Yang-Mills theory and which are also the starting point for Gompf's examples. The interaction between the complex geometry and the topology is particularly apparent in Kotschick's paper, and leading open problems, of detecting *rationality*, can be traced back to early work on algebraic surfaces.

The Dolgachev surfaces are also the starting point for the work described in the article of Kreck; the general setting is the relative theory, of 2-dimensional surfaces in 4-manifolds, and the Dolgachev manifolds appear as branched covers. Kreck's paper gives us an example of the application of the topological s-cobordism theorem, together with surgery theory, to a very concrete problem.

The paper of Johnson deals with a rather different facet of the topology of algebraic varieties; the structure of the fundamental group. There has been a good deal of activity in the last few years on the problems of describing what groups can occur as the fundamental groups of Kähler manifolds or of complex projective manifolds, with work of Johnson and Rees, Gromov, Toledo, Corlette, Goldman and Millson and others. A wide variety of techniques have been used, ranging from algebra to differential geometry and analysis. These questions are, at least vaguely, related to the techniques applied in defining differentiable invariants of complex surfaces, since the moduli spaces of unitary representations of the fundamental group of a compact Kähler manifold can be interpreted as moduli spaces of stable holomorphic vector bundles (compare, for example, the contribution of Okonek below).

[FM] Friedman, R. and Morgan, J.W. *Algebraic surfaces and 4-manifolds: some conjectures and speculations* Bull. Amer. Math. Soc. (New Series) **18** (1988) 1-18

[M] Mandelbaum,R. *Four-dimensional topology: an introduction* Bull. Amer. Math. Soc. (New Series) **2** (1980) 1-159

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Yang-Mills Invariants of Four-manifolds

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This article is based on three lectures given at the Symposium in Durham. In the first section we review the well-known analogies between Yang-Mills instantons over 4-manifolds and pseudo-holomorphic curves in almost-Kahler manifolds. The second section contains a rapid summary of the definition of invariants for smooth 4-manifolds using Yang-Mills moduli spaces, and of their main properties. In the third section we outline an extension of this theory, defining new invariants which we hope will have applications to connected sums of complex algebraic surfaces. Finally, in the fourth section, we take the opportunity to make some observations on pseudo-holomorphic curves and discuss the possibility of using linear analysis to construct symplectic submanifolds, in analogy with the Kodaira embedding theorem from complex geometry.

Section 1, Elliptic techniques in topological problems

The last ten years have seen the development and application of new techniques in the two fields of 4-manifold topology and symplectic geometry. There are striking parallels between these developments, both in detail and in general methodology. In the first case one is interested primarily in smooth, oriented 4-manifolds, and the problems of classification up to diffeomorphism. In the second case one is interested in, for example, problems of existence and uniqueness of symplectic structures (closed, nowhere degenerate, 2-forms). In each case the structure considered is locally standard : the only questions are global ones and it is reasonable to describe both subjects as "topological" in an extended sense of the word.
The new developments which we have in mind bring methods of geometry and analysis to bear on these topological questions. One introduces, as an auxiliary tool, some appropriate geometrical structure, which will have local invariants like curvature and torsion. In the case of symplectic manifolds this structure is a Riemannian metric adapted to the symplectic form or, equivalently, a compatible almost complex structure. Such a metric can appropriately be called *almost-Kahler* . In the other case one considers Riemannian metrics on 4-manifolds. With this structure fixed we study associated geometric objects : in the first case these are the *pseudo-holomorphic curves* in an almost complex manifold V (i.e. maps $f : \Sigma \to V$ from a

Riemann surface Σ with complex-linear derivative) ; in the other case the objects are the *Yang-Mills instantons* over a 4-manifold X (i.e. connections A on a principal bundle $P \to X$ with anti-self-dual curvature). In either case the objects can be viewed as the solutions of certain non-linear, elliptic, differential equations. Information about the original topological problem is extracted from properties of the solutions of these equations. In the symplectic case this strategy was first employed by Gromov [**15**], and the developments in both fields are instances of the use of "hard " techniques, in the terminology of Gromov [**16**].
The detailed analogies between these two set-ups are wide ranging. Among the most important are

(1) In each theory there is a "classical " or "integrable " case. On the one hand we can consider Kahler metrics on *complex* manifolds V, and their associated symplectic forms. Then the pseudo-holomorphic curves are the holomorphic curves in the ordinary sense. On the other hand we can consider the 4-manifolds obtained from complex projective surfaces, with Kahler metrics. Then, as we shall describe in Section 2 (c) below, the Yang-Mills instantons can be identified with certain holomorphic bundles over the complex surface. So in either theory our differential geometric objects can be described in algebro-geometric terms in these important cases.
(2) There is a fundamental integral formula in each case. The *area* of a compact pseudo-holomorphic curve equals its topological "degree " (the pairing of its fundamental class with the cohomology class of the symplectic form) ; and the *Yang-Mills energy* (mean- square of the curvature) of an instanton over a compact base manifold is a topological characteristic number of the bundle carrying the conection.
(3) Both theories are conformally invariant ; with regard to the structures on Σ and X respectively.
(4) The non-linear elliptic differential equations which arise in the two cases can have non-zero Fredholm indices. Thus the solutions are typically not isolated but are parametrised by moduli manifolds.
(5) Both theories enjoy strong links with Mathematical Physics (σ - models and gauge theories). A unified treatment of these developments from the point of view of quantum field theory has been given by Witten [**22**].
(6) Both theories exploit exploit special "low-dimensional" features – they are tied to the 2-dimensionality of Σ and the 4-dimensionality of X respectively.

There are many other points of contact between the theories. Notable among these are the developments in the two fields brought about through the magnificent work of Floer (see [**10**], and the articles on Floer's work in these Proceedings). Many of the developments in the two fields bear strongly on the *representation variety* W of conjugacy classes of representations of the fundamental group of a closed Riemann surface, which has a natural Kahler structure. For example the Casson invariant of a 3-manifold can be obtained from the intersection number of a pair of Lagrangian submanifolds in W. In a different setting we will encounter the space W in Section 2 (c) below, in our discussion of instantons over complex algebraic surfaces. It is intriguing that these representation spaces have also come to the fore recently in the Jones/Witten theory of invariants for knots and 3-manifolds (see the contributions of Atiyah, Hitchin, Kirby and Witten in the accompanying volume), and it seems quite likely that this points the way towards the possibility of obtaining some unified understanding of these different developments in Low-Dimensional Topology and Geometry.

Section 2, Yang-Mills invariants

(a) Definition. We will now describe how the Yang-Mills instantons yield invariants of certain smooth 4-manifolds. For more details see [**8**] or [**9**]. For brevity we will confine our discussion here to the gauge group SU(2), so we fix attention on a principal SU(2) bundle P over a compact, oriented Riemannian 4-manifold X. We will also assume that X is simply connected. The bundle P is determined up to isomorphism by the integer $k = < c_2(P), [X] >$, and if P is to support any anti-self-dual connection k must be non-negative, by the integral formula mentioned in (2) of Section 1. For each $k \geq 0$ we have a *moduli space* M_k of anti-self-dual connections on P modulo equivalence, and M_0 consists of a single point, representing the product connection on the trivial bundle.
Let A_0 be a solution of the instanton equations, i.e. $F^+(A_0) = 0$, where $F^+ = (1/2)(F + *F)$ denotes the self-dual part of the curvature. The curvature of another connection $A_0 + a$ can be written

$$F(A_0 + a) = F(A_0) + d_{A_0} a + a \wedge a,$$

where d_{A_0} is the coupled exterior derivative. Taking the self-dual part we get, in standard notation,

$$F^+(A_0 + a) = d^+_{A_0} a + (a \wedge a)^+.$$

The moduli space is obtained by dividing the solutions of this equation by the action of the "gauge group " $\mathcal{G} =$ Aut P. For small deformations a this division can be replaced by imposing the Coulomb gauge condition (provided the connection A_0 is irreducible)

$$d^*_{A_0} a = 0 \ ,$$

which defines a local transversal slice for the action of $\mathcal{G}$. Thus (assuming irreducibility) a neighbourhood of the point $[A_0]$ in the moduli space is given by the solutions of the differential equations

$$d^*_{A_0} a = 0$$
$$d^+_{A_0} a + (a \wedge a)^+ = 0.$$

These are non-linear, first order, equations; the non-linearity coming from the quadratic term $(a \wedge a)^+$. The linearisation about $a = 0$ can be written $\delta_{A_0} a = 0$, where $\delta_{A_0} = d^*_{A_0} \oplus d^+_{A_0}$ is a elliptic operator which plays the role in this four-dimensional situation of the Cauchy-Riemann operator in the theory of pseudo-holomorphic curves. The Fredholm index s = ind δ_{A_0} of this operator is given by the formula:

$$s = 8k - 3(1 + b^+(X) ,$$

in which $b^+(X)$ is the dimension of a maximal positive subspace for the intersection form on $H^2(X)$. The number s is the "virtual dimension" of the moduli space; more precisely, according to a theorem of Freed and Uhlenbeck [**11**], [**9**], for a generic Riemannian metric on X the part of the moduli space consisting of irreducible connections will be a smooth manifold of dimension s.

Let us now assume that $b^+(X)$ is strictly positive. Then it can be shown that for generic metrics and all $k \geq 1$ every instanton is irreducible. It is easy to see why b^+ enters here. A reducible anti-self-dual connection on P corresponds to an element c of $H^2(X; \mathbf{R})$ which is in the intersection of the integer lattice and the subspace $H^- \subset H^2$ consisting of classes represented by anti-self dual forms. The codimension of H^- is b^+, so if $b^+ > 0$ and H^- is in general position there are no non-zero classes in the intersection. On the same lines one can show that if $b^+ > 1$ then for generic 1 -parameter families of Riemannian metrics on X we do not encounter any non-trivial reducible connections.

We can now indicate how to define differential topological invariants of the underlying 4-manifold X. We introduce the space $\mathcal{B}^*$ of all irreducible connections on P, modulo equivalence. It is an infinite dimensional manifold and, under our assumptions the moduli space M_k is a submanifold of $\mathcal{B}^*$, for generic metrics on X. Roughly, the invariants we define are the pairings of the *fundamental homology class* of the moduli space with the cohomology of $\mathcal{B}^*$. To see that this is a reasonable strategy we have to consider the dependence of the definition on the Riemannian metric on X. The moduli space itself certainly depends on the choice of metric, so let us temporarily write $M_k(g)$ for the the moduli space defined with respect to a metric g. Suppose g_0 , g_1 are two generic metrics on X. We join them by a smooth path g_t; $t \in [0, 1]$ of metrics. If $b^+ > 1$ then, as explained above, we do not encounter any reducible connections so we can define

$$\mathcal{N} = \{ ([A], t) \in \mathcal{B}^* \times [0, 1] \mid [A] \in M_k(g_t)\}.$$

For a generic path g_t the space $\mathcal{N}$ is a manifold- with- boundary, the boundary consisting of the disjoint union of $M_k(g_0)$ and $M_k(g_1)$. Using the obvious projection from $\mathcal{N}$ to $\mathcal{B}^*$, we can regard $\mathcal{N}$ as giving a "homology" between the two moduli spaces.

This idea needs to be amplified in a number of ways. First we need to show that the moduli space is orientable (and to fix signs one must find a rule for choosing a definite orientation). Second we need to construct cohomology classes on $\mathcal{B}^*$. This second step is an exercise in algebraic topology. Fix a base point in X and let $\tilde{\mathcal{B}}$ be the $SO(3)$ bundle over $\mathcal{B}^*$ whose points represent equivalence classes of connections on a bundle which is trivialised over the base point. The space $\tilde{\mathcal{B}}$ is weak-homotopy equivalent to the space $Maps(X, BG)$ of based maps (of "degree" k) from X to the classifying space BG (which can be identified with $\mathbf{HP}^\infty$) of the structure group SU(2). One can show then that the rational cohomology of $\tilde{\mathcal{B}}$ is a polynomial algebra on 2-dimensional cohomology classes labelled by a basis for the 2-dimensional homology of X. That is, the cohomology is generated by the image of a natural map

$$\tilde{\mu} : H_2(X;\mathbf{Z}) \to H^2(\tilde{\mathcal{B}};\mathbf{Z}),$$

which is just the slant product in $Maps(X, BG) \times X$ with the 4-dimensional class pulled back from the generator of $H^4(BG)$ under the evaluation pairing $Maps(X, BG) \times X \to BG$. One can show further that this map $\tilde{\mu}$ descends to a map

$$\mu : H_2(X;\mathbf{Z}) \to H^2(\mathcal{B}^*;\mathbf{Z}),$$

and that the rational cohomology of $\mathcal{B}^*$ is freely generated as a ring by the image of this map and by a 4-dimensional class (the Pontryagin class of the fibration $\tilde{\mathcal{B}} \to \mathcal{B}^*$). The upshot of this algebro-topological excursion is that the *rational* cohomology classes of $\mathcal{B}^*$ are labelled by *polynomials* in the homology of X.
The third and most important step required to define invariants is to understand the compactness properties of the moduli space. If the moduli spaces were compact then they would carry fundamental homology classes in the usual way and there would be little extra to say. However in practice the moduli spaces are scarcely ever compact, but they do have natural compactifications. The compactification $\overline{M}_k$ of M_k is a subset of

$$M_k \ \cup \ M_{k-1} \times X \ \cup \ M_{k-2} \times s^2(X) \ \cup \ \ldots .$$

The topology is defined by a notion of convergence of the following kind. If $(x_1, \ldots, x_l)$ is a point in the symmetric product $s^l(X)$, a sequence $[A_n]$ in M_k converges to a limit $([A], (x_1, \ldots x_l)) \in M_{k-l} \times s^l(X)$ if the connections converge (up to equivalence) away from $x_1, \ldots, x_l$, and the energy densities $|F(A_n)|^2$ converge as measures to

$$|F(A)|^2 + 8\pi^2 \sum_{i=1}^{l} \delta_{x_i}.$$

The statement that the closure $\overline{M}_k$ of M_k in this topology is compact is essentially a handy formulation of analytical results of Uhlenbeck on Yang-Mills fields. This theory enters into our discussion of invariants because it can be used to show that

if the moduli space has even dimension, $s = 2d$ say, then for k such that $4k > (3b^+(X)+3)$ there is a natural pairing between the moduli space M_k and a product of cohomology classes $\mu(\alpha_1) \smile \mu(\alpha_2) \smile \cdots \smile \mu(\alpha_d)$, for any $\alpha_1, \ldots, \alpha_d$ in $H_2(X)$. We will refer to this range of values of as the "stable range" for k.
The cleanest conceptual definition of these pairings proceeds by extending the cohomology classes to the compactified space. For $l > 0$ and $c \in H^2(X)$ let $s^l(c) \in H^2(s^l(X))$ be the natural "symmetric sum " of copies of c. Then for α in $H^2(X)$ we let $\alpha^{(l)}$ be the class

$$\alpha^{(l)} = \pi_1^*(\mu(\alpha)) + \pi_2^*(s^l(c)) \in H^2(M_{k-l} \times s^l(X)\ ),$$

where c is the Poincaré dual of α. One then shows that, for any k, there is an extension $\overline{\mu}(\alpha)$ of $\mu(\alpha)$ to $H^2(\overline{M}_k)$, which agrees with $\alpha^{(l)}$ on $M_{k,l} \equiv \overline{M}_k \cap (M_{k-l} \times s^l(X))$. Consequently, for any $\alpha_1, \ldots, \alpha_d$ there is a class

$$\Pi = \overline{\mu}(\alpha_1) \smile \cdots \smile \overline{\mu}(\alpha_d) \in H^{2d}(\overline{M}_k).$$

Granted this we can define a pairing $< \Pi, [\overline{M}_k] >$ so long as the compactified space carries a fundamental homology class, and this fact follows from standard homology theory provided that the "strata " $M_{k,l}$ making up $\overline{M}_k$ have codimension 2 or more, for $l > 0$. But the dimension of $M_{k,l}$ is certainly bounded by that of $M_{k-l} \times s^l(X)$ which is :

(1) $\dim M_{k-l} + 4l = \dim M_k - 8l + 4l = \dim M_k - 4l$, if $l < k$;
(2) $\dim s_k(X) = 4k$, if $l = k$.

Since b^+ is odd the condition for $M_{k,k}$ to have codimension 2 is that $8k - 3(1 + b^+(X)) > 4k$, which is just the stable range condition stated above.
The disadvantage with this approach is that the only definition of the classes $\overline{\mu}(\alpha)$ known to the author is rather complicated (the main points in the definition are given in Chapter 7 of [**9**]). However the same pairing can be defined by a much more elementary, although less perspicuous, procedure. For a generic surface Σ in X the restriction of any ireducible anti-self-dual connection over X to Σ is again irreducible, so we get restriction maps :

$$r : M_j \to \mathcal{B}^*_\Sigma,$$

where $\mathcal{B}^*_\Sigma$ is the space of irreducible connections over Σ, modulo equivalence. If α is the fundamental class of Σ in $H_2(X)$ the cohomology class $\mu(\alpha)$ is pulled back from $\mathcal{B}^*_\Sigma$ by the restriction map. We choose a generic codimension 2 submanifold in this target space which represents the cohomology class, and let V_Σ be the pre-image of this in the moduli space. By abuse of notation we use the same symbol to denote subsets of all the different moduli spaces M_j (since they are all pulled back from the same representative over Σ). Let now $\Sigma_1, \ldots, \Sigma_d$ be surfaces in X, in general position, and write V_i for representatives V_{Σ_i}, as above. The crux of the matter is to show that, for k in the stable range, the intersection

$$M_k \cap V_1 \cap \cdots \cap V_d$$

is compact. We can then define the pairing to be the corresponding algebraic intersection number ; the number of points, counted with signs. The argument to establish this compactness is elementary, given two basic facts. First we can choose the V_i so that all intersections in all the moduli spaces are transverse (and the product connection is not in the closure of the V_i). Second, if $[A_n]$ is a sequence in $V_i \subset M_k$ which converges to $([A], x_1, \dots, x_l)$ in the sense considered above, and if none of the points x_j lies in Σ_i then the limit $[A]$ is in $V_i \subset M_j$. One then goes on to show that this intersection number is independent of the choice of Riemannian metric on X by intersecting $\mathcal{N}$ with the V_i. Similar arguments show that the intersection number is independent of the choice of V_i, and of the surfaces Σ_i, within their homology classes.

In sum, we have found new invariants of 4-manifolds which are multi-linear functions in the homology. We introduce the notation

$$\mathrm{Sym}^d_{X,R}$$

for the set of d-linear, symmetric, functions on $H_2(X;\mathbf{Z})$ with values in a ring R. Then we have

THEOREM 1. *Let X be a smooth, oriented, compact and simply connected 4-manifold with $b^+(X) = 2a+1$ for $a \geq 1$. For each k with $4k > (3b^+(X)+3)$ the map :*

$$q_k = q_{k,X} : ([\Sigma_1], \dots, [\Sigma_d]) \mapsto \sharp(V_1 \cap \cdots \cap V_d \cap M_k)$$

defines an element of $\mathrm{Sym}^d_{X,\mathbf{Z}}$, where $d = 4k - 3(1+a)$, which is (up to sign) a differential-topological invariant of X, natural with respect to orientation -preserving diffeomeorphisms.

We interpose a few remarks here. First, if $b^+ = 1$ one can still define invariants, but these have a more complicated form; see the article by Kotschick in these Proceedings. Second, it should be possible to extend the range of values of k for which invariants are defined. In a simple model case (where $b^+ = k = 1$) one knows how to introduce a boundary term to compensate for a codimension-1 stratum $M_{1,1}$, then one obtains the "Γ-invariant" of a 4-manifold. This approach has been extended in the Oxford D.Phil. thesis of K.C.Mong, and can probably be applied quite generally, although this has yet to be worked out in detail. A simpler procedure has been developed by J.W. Morgan, using components of the invariants for a connected sum $X \sharp\, r\overline{\mathbf{CP}}^2$, to define $q_{k,X}$ for values of k below the "stable range".