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Continuous and Discrete Modules

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PREFACE

The monograph addresses research mathematicians and graduate students interested in the module and representation theory of arbitary rings. It is primarily concerned with generalizations of injectivity and projectivity, and simultaneously with modules displaying good direct decomposition properties. Specifically, we study two classes of modules, named continuous and discrete. Both exhibit, in a dual sense, a generous supply of direct summands. The first class contains all injective modules, while the second one contains those projective modules which have a "good" direct sum decomposition.

Continuous, as the term is used here, is not related to continuity in the sense of topology and analysis. It is rather derived from the notion of a continuum. This usage originated with von Neumann's continuous geometries. These are analogues of projective geometries, except that they have no points, but instead a dimension function whose range is a continuum of real numbers. Just as most projective geometries can be coordinatized by simple artinian rings, most continuous geometries are coordinatized by non–noetherian continuous regular rings.

Utumi observed that continuous regular rings generalize self–injective regular rings. He extended the concept to arbitrary rings. Jeremy, Mohamed and Bouhy, and Goel and Jain generalized these ideas to modules.

The weaker notion of quasi–continuity appears now to be more fundamental. It asserts directly that the module inherits all direct sum decompositions from its injective hull (2.8). The important Theorem (2.31) ensures that uniqueness properties are inherited as well.

Another central result for quasi–continuous modules, (2.37), establishes a decomposition into a quasi–injective and a square–free part. This is a rare instance of a direct decomposition where both summands have, in different ways, better properties than the original module. It allows us to prove the exchange property for continuous modules (3.24), and the cancellation property for directly finite continuous modules (3.25).

Arbitrary families of orthogonal idempotents, of the endomorphism ring of a quasi–continuous module, lift modulo the ideal of endomorphisms with essential kernel (3.9). The endomorphism ring of a quasi–continuous module retains all the properties familiar from quasi–injective modules precisely if the module is actually continuous (3.15).

The dichotomy between projective and continuous geometries, namely that their dimension functions have discrete respectively continuum range, remains in effect for injective modules (cf. Goodearl and Boyle [76]; replace the dimension function by the finite rank function), and consequently for quasi–continuous modules. Noetherian rings are exactly the ones for which every injective or every quasi–continuous module is a direct sum of indecomposables. On the other hand, over "arbitrary" rings, the continuous structure is typical: an infinite direct sum of indecomposable (quasi–)continuous modules is (quasi–)continuous only in the presence of an ascending chain condition (2.13/3.16). A quasi–continuous module decomposes into indecomposables only in the presence of strong additional properties (2.22).

Concepts dual to those of (quasi–)continuity have been studied, under various names (notably (quasi–,semi–)perfect, (quasi–)dual continuous, stark supplementiert), by many authors. The usage of terminology is disturbingly inconsistent. We propose the new term "(quasi–)discrete", motivated by Oshiro's Theorem (4.15) that every such module is the direct sum of indecomposables.

This decomposition, which has strong uniqueness properties, reduces some proofs to counting arguments. Exchange and cancellation property, in particular, follow quite easily (4.19/20). Arbitrary families of orthogonal idempotents, of the endomorphism ring of a quasi–discrete module, lift modulo the ideal of endomorphisms with small image (5.9). Again, a quasi–discrete module is discrete precisely if the endomorphism ring exhibits all the familiar properties (5.4).

The converse question, when a direct sum of indecomposable quasi–discrete modules is quasi–discrete, has not yet received a fully satisfactory answer (cf. (4.48/49). The special cases of a finite direct sum (4.50), and a direct sum of local modules (4.53), are settled. For discrete modules over commutative noetherian rings, the complete answer is known (5.15/16), and requires elaborate arguments.

In spite of the dual nature of their definitions, and some analogies on an elementary level, (quasi–)continuous and (quasi–)discrete modules display striking dissimilarities as well: Continuity generalizes injectivity. The structure of quasi–continuous modules resembles that of their injective hulls. All injective/quasi–continuous modules are direct sums of indecomposables if and only if the ring is noetherian. A direct sum of quasi–continuous modules with full relative injectivity is quasi–continuous. On the other hand: Quasi–discrete modules are always direct sums of indecomposables. Discreteness generalizes projectivity if and only if the ring is perfect. A direct sum of quasi–discrete modules with full relative projectivity need not be quasi–discrete.

These dissimiliarities can be traced to the fact that every module is complemented, and hence possesses an injective hull, while most modules are not supplemented (4.41), and have no projective cover. Modules which are just supplemented, have interesting properties of their own, and were studied by many people (cf. Appendix, Section 1 and 2). Quasi–discrete modules are supplemented (4.8), and thus constitute a much more restrictive class than quasi–continuous ones.

Chapter 1 is of preliminary nature. It summarizes facts about relative injectivity, and proves the exchange and cancellation properties for injective modules. It also develops a general technique for constructing direct sum decompositions, and derives the decomposition of an injective module into a directly finite and a purely infinite part. Analogous results on relative projectivity are collected at the beginning of Section 4 of Chapter 4. More details concerning the arrangement of the material may be obtained from the table of content.

We have attempted to provide a complete and up to date account of the subject. The exposition is self contained, except that a few well known and highly technical results which are readily accessible in the literature, are quoted without proof. All undefined concepts can be found in Anderson and Fuller [73]. In the comments at the end of each chapter, we try to trace the origin of some of the main ideas. Section 6 of the Appendix lists a number of open questions.

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The main portion of this work was done while the first author was on sabbatical leave from Kuwait University, at the second author's institution, McMaster University. The authors acknowledge partial financial support by Kuwait University and by the National Science and Engineering Research Council of Canada.

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