CHAPTER 1

INJECTIVITY AND RELATED CONCEPTS

In this chapter we discuss injectivity, quasi–injectivity and relative injectivity, with emphasis on those properties which are used later on in the book. We start by listing some of the well known fundamental properties of injective modules which can be found in Anderson and Fuller [73] or Sharpe and Vamos [72].

A module E is *injective* if it satisfies any of the equivalent conditions:

(1) For every module A and any submodule X of A every homomorphism $X \longrightarrow E$ can be extended to a homomorphism $A \longrightarrow E$;

(2) (Baer's Criterion) Every homomorphism of a right ideal I of R to E can be extended to a homomorphism of R to E;

(3) For any module M every monomorphism $E \rightarrowtail M$ splits;

(4) E has no proper essential extensions.

Every module M has a minimal injective extension, which is at the same time a maximal essential extension of M; such an extension is unique up to isomorphism and is called the *injective hull* of M. The injective hull of M will be denoted by E(M).

1. A–INJECTIVE MODULES

__Definition 1.1__. Let A be an R–module. A module N is said to be A–*injective* if for every submodule X of A, any homomorphism $\varphi: X \longrightarrow N$ can be extended to a homomorphism $\psi : A \longrightarrow N$.

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "\varphi"'] & A \arrow[dl, dashed, "\psi"] \\ N & \end{tikzcd}$$

The following is an immediate consequence.

__Lemma 1.2__. *If N is A–injective, then any monomorphism* $N \overset{f}{\rightarrowtail} A$ *splits. If, in addition, A is indecomposable, then f is an isomorphism.* □

__Proposition 1.3__. *Let N be an A–injective module. If* $B \le A$, *then N is B–injective and A/B–injective.*

PROOF. It is obvious that N is B–injective. Let X/B be a submodule of A/B, and $\varphi : X/B \longrightarrow N$ be a homomorphism. Let π denote the natural homomorphism of A onto A/B and $\pi' = \pi|_X$. Since N is A–injective, there exists a homomorphism $\theta : A \longrightarrow N$ that extends $\varphi\pi'$. Now

$$\theta B = \varphi\pi' B = \varphi(0) = 0.$$

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d, "\pi'"'] & A \arrow[d, "\pi"] \arrow[ddl, dashed, "\theta"'] \\
X/B \arrow[r, tail] \arrow[d, "\varphi"'] & A/B \arrow[dl, dashed, "\psi"] \\
N &
\end{tikzcd}$$

Hence $\operatorname{Ker} \pi \leq \operatorname{Ker} \theta$, and consequently there exists $\psi : A/B \longrightarrow N$ such that $\psi\pi = \theta$. For every $x \in X$

$$\psi(x + B) = \psi\pi(x) = \theta(x) = \varphi\pi'(x) = \varphi(x + B).$$

Thus ψ extends φ, and therefore N is A/B injective. □

The following proposition may be viewed as a generalization of Baer's Criterion.

__Proposition 1.4__. *A module* N *is A–injective if and only if* N *is* aR*–injective for every* $a \in A$.

PROOF. The "only if" part follows by the preceding proposition.

Conversely, assume that N is aR–injective for every $a \in A$. Let $X \leq A$ and $\varphi : X \longrightarrow N$ be a homomorphism. By Zorn's Lemma, we can find a pair (B, ψ) maximal with the properties $X \leq B \leq A$ and $\psi : B \longrightarrow N$ is a homomorphism which extends φ. It is clear that $B \leq^e A$. Suppose that $B \neq A$ and consider an element $a \in A - B$. Let $K = \{r \in R : ar \in B\}$; then it is clear that $aK \neq 0$. Define $\mu : aK \longrightarrow N$ by $\mu(ak) = \psi(ak)$. Then by assumption μ can be extended to $\nu : aR \longrightarrow N$.

Now define $\chi : B + aR \longrightarrow N$ by $\chi(b + ar) = \psi(b) + \nu(ar)$. Then χ is well defined, since if $b + ar = 0$, then $r \in K$ and so

$$\psi(b) + \nu(ar) = \psi(b) + \mu(ar) = \psi(b) + \psi(ar) = \psi(b + ar) = 0.$$

But then the pair $(B + aR, \chi)$ contradicts the maximality of (B, ψ). Hence $B = A$, and $\psi : A \longrightarrow N$ extends φ. □

__Proposition 1.5__. *A module* N *is* $(\bigoplus_{i \in I} A_i)$*–injective if and only if* N *is* A_i*–injective for every* $i \in I$.

PROOF. Assume that N is A_i–injective for all $i \in I$. Let $A = \bigoplus_{i \in I} A_i$, $X \leq A$ and consider a homomorphism $\varphi : X \longrightarrow N$. We may assume, by Zorn's Lemma, that φ cannot be extended to a homomorphism $X' \longrightarrow N$ for any submodule X' of A which contains X properly. Then $X \leq^e A$. We claim that $X = A$. Suppose not. Then there

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exist $j \in I$ and $a \in A_j$ such that $a \notin X$. Since N is A_j–injective, N is aR–injective by Proposition 1.3. By an argument similar to that given in Proposition 1.4, we can extended φ to a homomorphism $\psi : X + aR \longrightarrow N$, which contradicts the maximality of φ. This proves our claim, and hence N is A–injective.

The converse follows by Proposition 1.3. □

The same proof as for injective modules yields the following

Proposition 1.6. $\prod_{\alpha \in \Lambda} M_\alpha$ *is A–injective if and only if* M_α *is A–injective for every* $\alpha \in \Lambda$. □

Next we investigate the A–injectivity of direct sums.

Theorem 1.7. *The following are equivalent for a family of modules* $\{M_\alpha : \alpha \in \Lambda\}$:

(1) $\bigoplus_{\alpha \in \Lambda} M_\alpha$ *is A–injective;*

(2) $\bigoplus_{i \in I} M_i$ *is A–injective for every countable subset* $I \subseteq \Lambda$;

(3) M_α *is A–injective for every* $\alpha \in \Lambda$, *and for every choice of* $m_i \in M_{\alpha_i}$ $(i \in \mathbb{N})$ *for distinct* $\alpha_i \in \Lambda$ *such that* $\bigcap_{i=1}^{\infty} m_i^o \geq a^o$ *for some* $a \in A$, *the ascending sequence* $\bigcap_{i \geq n} m_i^o$ $(n \in \mathbb{N})$ *becomes stationary.*

PROOF. (1) ⇒ (2) follows by Proposition 1.6.

(2) ⇒ (3): Proposition 1.6 implies that M_α is a A–injective for every $\alpha \in A$. Consider the element $x = (m_i) \in \prod_{i=1}^{\infty} M_{\alpha_i}$. The mapping φ: $ar \to xr$ is a well defined homomorphism from aR to $\prod_{i=1}^{\infty} M_{\alpha_i}$. Let $I = \bigcup_{n=1}^{\infty} (\bigcap_{i \geq n} m_i^o)$, and let $\bar{\varphi}$ denote the restriction of φ to aI. Then $\bar{\varphi}$ is a homomorphism of aI into $\bigoplus_{i=1}^{\infty} M_{\alpha_i}$. Since $\bigoplus_{i=1}^{\infty} M_{\alpha_i}$ is A–injective and hence aR–injective, $\bar{\varphi}$ extends to ψ: $aR \longrightarrow \bigoplus_{i=1}^{\infty} M_{\alpha_i}$. Then

$$xI = \bar{\varphi}(aI) = \psi(aI) \leq \psi(aR) = \psi(a)R \leq \bigoplus_{i \in F} M_{\alpha_i},$$

where F is a finite subset of $\mathbb{N}$. Let $F = \{1,2,\dots,k-1\}$. Then $m_i I = 0$ for $i \geq k$ and hence $I = \bigcap_{i \geq k} m_i^o$. Therefore the sequence $\bigcap_{i \geq n} m_i^o$ $(n \in \mathbb{N})$ becomes stationary.

(3) ⇒ (1): By way of contradiction, assume that $\bigoplus_{\alpha \in \Lambda} M_\alpha$ is not A–injective. Then by

Proposition 1.4, $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is not aR–injective for some $a\in A$. Hence there exists a right ideal K of R and a homomorphism f: $aK \longrightarrow \bigoplus_{\alpha\in\Lambda} M_\alpha$ such that f cannot be extended to aR. Since $\bigoplus_{\alpha\in F} M_\alpha$ is A–injective for all finite subsets $F \subseteq \Lambda$ by Proposition 1.6, $f(aK) \nsubseteq \bigoplus_{\alpha\in F} M_\alpha$ for any finite subset $F \subseteq \Lambda$. However f can

$$\begin{tikzcd} aK \arrow[r, tail] \arrow[d, "f"'] & aR \arrow[ddl, dashed, "g"] \\ \bigoplus M_\alpha \arrow[d, tail] & \\ \prod M_\alpha & \end{tikzcd}$$

be extended to $g : aR \longrightarrow \prod_{\alpha\in\Lambda} M_\alpha$ since $\prod_{\alpha\in\Lambda} M_\alpha$ is A–injective. Let $m = g(a)$. Then it is clear that $a^o \leq m^o = \bigcap_{\alpha\in\Lambda} m_\alpha^o$ where m_α is the α–component of $m\in \prod_{\alpha\in\Lambda} M_\alpha$. Then Let $S_k = \{\alpha\in\Lambda : m_\alpha k \neq 0\}$, $k\in K$. Then S_k is a finite subset of Λ for every $k\in K$. However $I = \bigcup_{k\in K} S_k$ is not finite since $mK = f(aK) \nsubseteq \bigoplus_{\alpha\in F} M_\alpha$ for any finite subset $F \subseteq \Lambda$. By induction we select elements $k_i\in K$ ($i\in\mathbb{N}$) and indices $\alpha_j\in\Lambda$ such that $\alpha_j\in S_{k_j}$ and $\alpha_j \notin \bigcup_{i=1}^{j-1} S_{k_i}$. Let m_i denote the α_i–component of m. Then $a^o \leq \bigcap_{i=1}^{\infty} m_i^o$ and the sequence $\bigcap_{i\geq n} m_i^o$ ($n\in\mathbb{N}$) is strictly increasing, which is a contraction to our assumption. Therefore $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is A–injective. □

<u>Corollary 1.8</u>. $\bigoplus_{i=1}^{\infty} M_i$ *is* A*–injective if and only if* M_i *is* A*–injective for every* $i\in\mathbb{N}$, *and for every choice* $m_i\in M_i$ *such that* $\bigcap_{i=1}^{\infty} m_i^o \geq a^o$ *for some* $a\in A$, *the ascending sequence* $\bigcap_{i\geq n} m_i^o$ ($n\in\mathbb{N}$) *becomes stationary.* □

Motivated by these results and later applications, we introduce the following three chain conditions on a ring R relative to a given family of R–modules $\{M_\alpha : \alpha\in\Lambda\}$:

(A_1) For every choice of distinct $\alpha_i\in\Lambda$ ($i\in\mathbb{N}$) and $m_i\in M_{\alpha_i}$ the ascending sequence $\bigcap_{i\geq n} m_i^o$ ($n\in\mathbb{N}$) becomes stationary;

(A_2) Fore every choice of $x\in M_\alpha$ ($\alpha\in\Lambda$) and $m_i\in M_{\alpha_i}$ for distinct $\alpha_i\in\Lambda$ ($i\in\mathbb{N}$) such that $m_i^o \geq x^o$, the ascending sequence $\bigcap_{i\geq n} m_i^o$ ($n\in\mathbb{N}$) becomes stationary;

(A_3) For every choice of distinct $\alpha_i \in \Lambda$ $(i \in \mathbb{N})$ and $m_i \in M_{\alpha_i}$, if the sequence m_i^o is ascending, then it becomes stationary.

It is clear that (A_1) implies (A_2) and (A_2) implies (A_3). No other implication holds as we shall see by the end of this section.

The following is a consequence of Proposition 1.6 and Theorem 1.7.

__Proposition 1.9__. *Let* $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$. *Then* $M(\Lambda - \alpha)$ *is* M_α*–injective for every* $\alpha \in \Lambda$ *if and only if* M_α *is* M_β*–injective for all* $\alpha \neq \beta \in \Lambda$ *and* (A_2) *holds.*

□

By Proposition 1.6, a direct product of injective modules is injective, and hence a finite direct sum of injective modules is injective. The following proposition, which deals with injectivity of arbitrary direct sums, is an immediate consequence of Theorem 1.7.

__Proposition 1.10__. $\bigoplus_{\alpha\in\Lambda} M_\alpha$ *is injective if and only if each* M_α *is injective and* (A_1) *holds.*

□

__Theorem 1.11__. *The direct sum of any family of* A*–injective modules is* A*–injective if and only if every cyclic (or finitely generated) submodule of* A *is noetherian. In particular, the direct sum of every family of injective* R*–modules is injective if and only if* R *is right noetherian.*

PROOF. Assume that aR is noetherian for every $a \in A$, and consider a direct sum $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ of A–injective modules M_α. Let $B \leq aR$ and $\varphi : B \longrightarrow M$ be a homomorphism. Since B is finitely generated, $\varphi(B) \leq \bigoplus_{\alpha\in F} M_\alpha$ for a finite subset $F \subseteq \Lambda$. Then φ can be extended to $\psi : aR \longrightarrow \bigoplus_{\alpha\in F} M_\alpha$, since $\bigoplus_{\alpha\in F} M_\alpha$ is A–injective by Proposition 1.6. Hence the A–injectivity of M follows by Proposition 1.4.

Conversely, assume that the direct sum of any family of A–injective modules is A–injective. Let a be an arbitrary element of A. We prove that aR is right noetherian by showing that any ascending sequence

$$a^o = B_o \leq B_1 \leq B_2 \leq \ldots$$

of right ideals of R is ultimately stationary. Let $M_i = E(R/B_i)$, $i \in \mathbb{N}$. Since each M_i is trivially A–injective, $\overset{\infty}{\underset{i=1}{\oplus}} M_i$ is A–injective by assumption. Consider the set of elements $\{m_i = 1 + B_i : i \in \mathbb{N}\}$. The A–injectivity of $\overset{\infty}{\underset{i=1}{\oplus}} M_i$ implies, by Corollary 1.8, that the ascending sequence $\underset{i \geq n}{\cap} m_i^o$ $(n \in \mathbb{N})$ becomes stationary. As $m_i^o = B_i$ for every $i \in \mathbb{N}$,

$$B_n = m_n^o = \underset{i \geq n}{\cap} m_i^o.$$

Hence the sequence $B_1 \leq B_2 \leq \ldots$ becomes stationary, and consequently aR is noetherian.

The last statement is obvious. □

We conclude this section by listing examples which seperate the ascending chain conditions (A_1), (A_2) and (A_3). Each of these examples is of the type $\underset{i \in \mathbb{N}}{\oplus} M_i$ with indecomposable injective M_i.

Examples 1.12. (1) Let R be any commutative domain, and let K be its quotient field. If we take $M_i = K$ $(i \in \mathbb{N})$, then $\underset{i \in \mathbb{N}}{\oplus} M_i$ is injective, hence (A_1) holds. However R is not necessarily noetherian.

(2) Let $R = \underset{i \in \mathbb{N}}{\Pi} K_i$, a product of fields; and $M_i = K_i$. Here $\underset{i \in \mathbb{N}}{\oplus} M_i$ is semisimple, hence it is obvious that (A_2) holds. Since $E(\underset{i \in \mathbb{N}}{\oplus} M_i) = \underset{i \in \mathbb{N}}{\Pi} K_i$, $\underset{i \in \mathbb{N}}{\oplus} M_i$ is not injective, hence (A_1) does not hold.

(3) Let R be any (left and right) perfect ring such that $E(R_R)$ is projective but $E({}_RR)$ is not (for the existence of such a ring, see Müller [68]). Let M be a direct sum of countably many copies of $E(R_R)$. Then M is not quasi–injective by (Yamagata [74], Lemma 3.1). Since $E(R_R)$ is projective, it is a finite direct sum of indecomposables; so $M = \underset{i \in \mathbb{N}}{\oplus} M_i$, with each M_i indecomposable injective. That $\underset{i \in \mathbb{N}}{\oplus} M_i$ does not have (A_2) follows by Proposition 1.9 (see also Proposition 1.18). – Proposition 2.24 (see Definition 2.23).

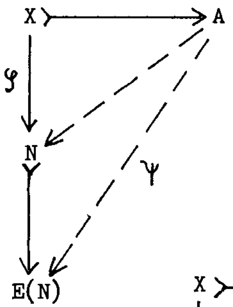
(4) For an incidence where even (A_3) fails, consider any local generalized quasi–Frobenius ring, and let $M_i = R (i \in \mathbb{N})$. Then $\underset{i \in \mathbb{N}}{\oplus} M_i$ has (A_3) if and only if it is locally–semi–T–nilpotent (Proposition 2.24), consequently R is perfect and hence quasi–Frobenius. An explicit example of a local generalized quasi–Frobenius ring which is not quasi–Frobenius is $R = \hat{\mathbb{Z}}_p \ltimes C_p^\infty$, the split extension of the ring $\hat{\mathbb{Z}}_p$ of p–adic integers by the Prüfer group C_p^∞.

2. QUASI–INJECTIVE MODULES

A module Q is called *quasi–injective* if it is Q–injective. Quasi–injective modules are closely related to their injective hulls. We investigate this relationship in a more general setting.

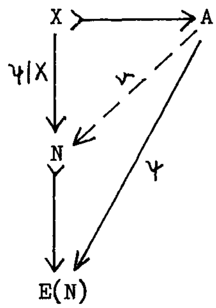
<u>Lemma 1.13</u>. *A module* N *is* A*-injective if and only if* $\psi A \leq N$ *for every* $\psi \in \mathrm{Hom}\,(E(A), E(N))$.

PROOF. Since E(N) is injective, it is enough to consider $\psi \in \mathrm{Hom}\,(A, E(N))$.



"If": Let $X \leq A$ and $\varphi : X \longrightarrow N$ be a homomorphism. Since E(N) is injective, φ can be extended to $\psi : A \longrightarrow E(N)$. By assumption $\psi A \leq N$, and hence $\psi : A \longrightarrow N$ extends φ. Therefore N is A–injective.

"Only if" : Let $X = \{a \in A : \psi(a) \in N\}$. Since N is A–injective, $\psi|_X$ can be extended to $\nu : A \longrightarrow N$. We claim that $N \cap (\nu - \psi)A = 0$. Indeed, let $n \in N$ and $a \in A$ be such that $n = (\nu - \psi)(a)$. Then $\psi(a) = \nu(a) - n \in N$, and consequently $a \in X$. Then $n = \nu(a) - \psi(a) = \psi(a) - \psi(a) = 0$.



Therefore $N \cap (\nu - \psi)A = 0$, and hence $(\nu - \psi)A = 0$ as $N \leq^e E(N)$. Hence $\psi A = \nu A \leq N$. □

<u>Corollary 1.14</u>. *A module* Q *is quasi–injective if and only if* $fQ \leq Q$ *for every* $f \in \mathrm{End}\ E(Q)$. □

<u>Corollary 1.15</u>. *Every module* M *has a minimal quasi–injective extension, which is unique up to isomorphism.*

PROOF. Let $Q(M) = (\mathrm{End}\ E(M))(M)$. Then it is obvious that Q(M) satisfies the required conditions. □

Lemma 1.13 has also the following

<u>Corollary 1.16</u>. *Let* A *and* B *be relatively injective* (i.e. A is B–injective and B is A–injective). *If* $E(A) \cong E(B)$, *then* $A \cong B$; *in fact any isomorphism* $E(A) \longrightarrow E(B)$ *restricts to an isomorphism* $A \longrightarrow B$; *in addition* A *and* B *are quasi–injective.*

PROOF. Let $g : E(A) \longrightarrow E(B)$ be an isomorphism. Since B is A–injective, $gA \leq B$ by Lemma 1.13. Similarly $g^{-1}B \leq A$. Hence

$$B = (gg^{-1})B = g(g^{-1}B) \leq gA \leq B.$$

Consequently $gA = B$, and therefore $g|_A : A \longrightarrow B$ is an isomorphism.

Since A is B–injective and $B \cong A$, A is A–injective, that is A is quasi–injective. □

The following is an immediate consequence of Propositions 1.5 and 1.6.

<u>Proposition 1.17</u>. *$M_1 \oplus M_2$ is quasi–injective if and only if M_i is M_j–injective* ($i,j = 1,2$). *In particular, a summand of a quasi–injective module is quasi–injective.* □

Now consider an arbitrary direct sum $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$. It is clear from the preceding proposition that "M_α is M_β–injective for all $\alpha, \beta\in\Lambda$" is a necessary condition for M to be quasi–injective. The following result (which is analogous to Proposition 1.10) shows that this condition is also sufficient in the presence of condition (A_2).

<u>Proposition 1.18</u>. *The following are equivalent for a direct sum decomposition of a module* $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$:

(1) M *is quasi–injective;*

(2) M_α *is quasi–injective and* $M(\Lambda - \alpha)$ *is* M_α*–injective for every* $\alpha\in\Lambda$;

(3) M_α *is* M_β*–injective for all* $\alpha, \beta\in\Lambda$ *and* (A_2) *holds.*

PROOF. Using Propositions 1.9 and 1.17, the proof is straightforward. □

<u>Corollary 1.19</u>. *$\bigoplus_{i=1}^{n} M_i$ is quasi–injective if and only if M_i is M_j–injective* ($i, j = 1,2,\ldots,n$). *M^n is quasi–injective if and only if M is quasi–injective.* □

3. EXCHANGE AND CANCELLATION PROPERTIES

In this section, we prove that every (quasi–)injective module has the exchange property, and an injective module has the cancellation property if and only if it is directly finite. (The terms are defined below).

Definition 1.20. A module M is said to have the *(finite) exchange property* if for any (finite) index set I, whenever $M \oplus N = \bigoplus_{i \in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\bigoplus_{i \in I} B_i)$ for submodules $B_i \leq A_i$.

It is fairly easy to show that the (finite) exchange property is inherited by summands and finite direct sums (see Lemma 3.20).

Theorem 1.21. *Every quasi–injective module* M *has the exchange property.*

PROOF. Let $A = M \oplus N = \bigoplus_{i \in I} A_i$. Let $X_i = A_i \cap N$ and $X = \bigoplus_{i \in I} X_i$. By Zorn's Lemma, we can find $B \leq A$ maximal with respect to the following properties:

(i) $B = \bigoplus_{i \in I} B_i$ with $X_i \leq B_i \leq A_i$,

(ii) $M \cap B = 0$.

We claim that $A = M \oplus B$. Our claim will hold if we show that $\overline{M} \leq^e \overline{A}$ and $\overline{M} \subset^{\oplus} \overline{A}$ (where $\overline{Y}$ denotes the image of $Y \leq A$ under the natural homomorphism $A \longrightarrow A/B$). We start by showing $\overline{M} \cap \overline{A}_j \leq^e \overline{A}_j$ for every $j \in I$. Let D be an arbitrary submodule of A_j such that $B_j < D$. Then

$$B < D + B = D \oplus (\bigoplus_{i \neq j} B_i).$$

Maximality of B then implies $M \cap (D + B) \neq 0$. Since $M \cap B = 0$, $M \cap (D + B) \nleq B$. Hence

$$(\overline{M} \cap \overline{A}_j) \cap \overline{D} = \overline{M} \cap \overline{D} \neq 0.$$

Hence $\overline{M} \cap \overline{A}_j \leq^e \overline{A}_j$ for all $j \in I$. Consequently

$$\bigoplus_{j \in I} (\overline{M} \cap \overline{A}_j) \leq^e \bigoplus_{j \in I} \overline{A}_j = \overline{A}.$$

Therefore $\overline{M} \leq^e \overline{A}$.

If M is injective, then $\overline{M}$ is also injective (since $\overline{M} = (M \oplus B)/B \cong M$) and $\overline{M} \subset^{\oplus} \overline{A}$ follows trivially. For a quasi– injective module we have the following additional argument: Let π be the projection $M \oplus N \longrightarrow\!\!> M$. The restriction of π to A_i has kernel X_i, and hence A_i/X_i is isomorphic to a submodule of M. Since M is M–injective, M is A_i/X_i–injective by Proposition 1.3. As $A/X \cong \oplus A_i/X_i$, we get by Proposition 1.5 that M is A/X–injective; hence M is A/B–injective by Proposition 1.3. Since $\overline{M} \cong M$, $\overline{M}$ is $\overline{A}$–injective and therefore $\overline{M} \subset^{\oplus} \overline{A}$ by Lemma 1.2. □

Definition 1.22. A module M is said to have the *cancellation property* if whenever $M \oplus X \cong M \oplus Y$, then $X \cong Y$. M is said to have the *internal cancellation property* if whenever $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$.

Proposition 1.23. *Let* M *be a module with the finite exchange property. Then* M *has the cancellation property if and only if* M *has the internal cancellation property.*

PROOF. "Only if" : Let $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$. Then

$$M \oplus B_1 = A_2 \oplus B_2 \oplus B_1 \cong A_1 \oplus B_2 \oplus B_1 = M \oplus B_2.$$

Hence $B_1 \cong B_2$. (This direction of the proof does not need the finite exchange property on M).

"If" : Let $M \oplus X = N \oplus Y$ with $M \cong N$. By the finite exchange property we get $M \oplus X = M \oplus N' \oplus Y'$ such that $N' \leq N$ and $Y' \leq Y$. Then $X \cong N' \oplus Y'$. It is also clear that $N' \subset^{\oplus} N$ and $Y' \subset^{\oplus} Y$; write $N = N' \oplus N''$ and $Y = Y' \oplus Y''$. Then

$$M \oplus N' \oplus Y' = M \oplus X = N \oplus Y = N' \oplus N'' \oplus Y' \oplus Y''.$$

Hence $M \cong N'' \oplus Y''$, and therefore

$$N'' \oplus Y'' \cong M \cong N = N'' \oplus N'.$$

Since M has the internal cancellation property, $N' \cong Y''$; hence

$$X \cong N' \oplus Y' \cong Y'' \oplus Y' = Y.$$ □

Definition 1.24. A module D is called *directly finite* if D is not isomorphic to a proper summand of itself.

It is clear that a summand of a directly finite module is again directly finite.

The following is a characterization of directly finite modules via their endomorphism rings.

Proposition 1.25. *A module* D *is directly finite if and only if* $fg = 1$ *implies that* $gf = 1$ *for all* $f, g \in \text{End } M$.

PROOF. Assume that $fg = 1$ for some $f, g \in \text{End } M$. Then $D = gD \oplus \text{Ker} f$. Since $gD \cong D$ and D is directly finite, $\text{Ker} f = 0$. However f is onto, and therefore f is an automorphism of D. Hence $gf = 1$.

Conversely, assume the condition and let $D = B \oplus C$ with $B \cong D$. Let $\varphi : D \longrightarrow B$ be an isomorphism. Define φ^* as φ^{-1} on B and 0 on C. Then $\varphi^* \varphi = 1$ and hence $\varphi \varphi^* = 1$. It then follows that φ^* is a monomorphism and hence $C = 0$.

□

Lemma 1.26. *If* M *is not directly finite, then* $X^{(\mathbb{N})}$ *embeds in* M *for some non–zero module* X.