Cambridge University Press
978-0-521-39849-7 - Groups St Andrews 1989, Volume 1
Edited by C. M. Campbell and E. F. Robertson
Excerpt
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TRIPLY FACTORIZED GROUPS

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Let the group G = AB be the product of two subgroups A and B. If N is a normal subgroup of G, then it is easy to see that the factorizer $X(N) = AN \cap BN$ has the triple factorization

$$X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)$$

(see for instance [1]). Therefore it is of interest to consider triply factorized groups of the form

$$G = AB = AK = BK \text{ where K is a normal subgroup of G.}$$

If the group theoretical property $\mathfrak{X}$ is inherited by epimorphic images and extensions, then clearly the triply factorized group G is an $\mathfrak{X}$-group whenever A, B and K are $\mathfrak{X}$-groups. This will usually fail to be the case if $\mathfrak{X}$ is not extension inherited and is for instance some nilpotency or supersolubility requirement. This is most strikingly seen from the following examples. (A survey on results about general infinite factorized groups can be found in [3].)

1. Some examples

In [23] Sysak has given examples of triply factorized groups G = AB = AK = BK where A, B and K are abelian and K is normal in G. Among these there are some which are not nilpotent in any reasonable sense.

1.1 General Construction. Let R be a radical ring and let A be the set R with operation

$$r \circ s = r + s + rs \text{ for every } r, s \in R.$$

Then A is a group which operates on the additive group $M = R^+$ of R via

$$m^r = m \circ r - r = m + mr \quad \text{for every } m \in M \text{ and } r \in A.$$

Consider the semi-direct product

$$G = A \ltimes M = \{(r,m) \mid r \in A, m \in R\}.$$

Define the following subgroups of G:

$\{(r,0) \mid r \in A\}$,

$\{(0,m) \mid m \in R\} \subseteq R^+$,

$\{(r,r) \mid r \in A\}$.

If we identify these with A, M and B respectively, then the following holds:

$1 = B \cap M = A \cap M = A \cap B$.

Multiplication in G is given by

$(r,s)(r',s') = (r + r' + rr', s + s' + sr')$.

In particular we have

$(r,r)(r',0) = (r + r' + rr', r + rr') = (0,m) \in M$,

so that $r + r' + rr' = 0$ and $r' = -m$. This shows that $M \subseteq BA$, and similarly $A \subseteq BM$. Hence $G = AM \subseteq BM \subseteq BA$. We have shown:

$G = AM = BM = AB$, $A \cap M = B \cap M = A \cap B = 1$, A, B and M abelian.

Using the above multiplication rule for the semidirect product we compute

$(0,-s)(r,0)(0,s) = (0,-s)(r,s) = (r,-sr) = (r,0)(0,-sr)$,

so that

$(0,-sr) = [(r,0), (0,s)]$.

We suppose now that the radical ring R satisfies the following additional requirement:

For every $r \neq 0$ there exists an element s in R with $sr \neq 0$. (*)

Assume there exists an element (r,0) in the core A_G of A with $r \neq 0$. Then we have that

$1 \neq (0,-sr) = [(r,0), (0,s)] \in A_G \subseteq A$.

This contradiction shows $A_G = 1$, and similarly $B_G = 1$. Then also $A \cap Z(G)$ and $B \cap Z(G)$ are trivial, so that it follows from the following lemma that $Z(G) = 1$. In particular G is not hypercentral.

Lemma. *If the group* $G = AB$ *is the product of two abelian subgroups* A *and* B, *then* $Z(G) = (A \cap Z(G))(B \cap Z(G))$.

To prove this let $c = ab^{-1}$ with $a \in A$ and $b \in B$ be a non-trivial element of $Z(G)$. Let $s = a^*b^*$ be an arbitrary element of G with $a^* \in A$ and $b^* \in B$. Then

$$[a,s] = [a, a^*b^*] = [a,b^*][a,a^*]^{b^*} = [a,b^*] = [cb,b^*]$$

$$= [c,b^*]^b[b,b^*] = [c,b^*]^b = 1.$$

Therefore $a \neq 1$ is contained in $Z(G)$ and similarly also b is in $Z(G)$. This proves the lemma.

1.2 Example. Let $p \neq 2$ be a prime, and let R be the ring of all rational numbers of the form u/v where u and v are integers with $v \neq 0$, p divides u, but p does not divide v. Then R is a radical ring and clearly it also satisfies condition (*). Clearly $M = R^+$ as a subgroup of $\mathbf{Q}^+$ is torsion-free of Prüfer rank 1. The groups A and B are free abelian of countable infinite rank. Hence also the group G is torsion-free. It can be shown that G is not locally nilpotent and not even locally polycyclic.

2. Finite groups

An epimorphism inherited class of finite groups $\mathfrak{F}$ is a saturated formation if for every finite group G the following holds: (i) $G/(N \cap M)$ is an $\mathfrak{F}$-group whenever G/N and G/M are $\mathfrak{F}$-groups and (ii) G is an $\mathfrak{F}$-group whenever the Frattini factor group $G/\mathrm{Frat}\, G$ is an $\mathfrak{F}$-group. Examples for saturated formations are the classes of finite nilpotent and finite supersoluble groups.

Theorem 2.1. *Let* $\mathfrak{F}$ *be a saturated formation of finite groups, and let the group* $G = AB = AK = BK$ *be the product of three subgroups* A, B *and* K, *where* K *is normal in* G. *If* A *and* B *are* $\mathfrak{F}$*-groups and* K *is nilpotent, then* G *is an* $\mathfrak{F}$*-group.*

Proof. Assume that Theorem 2.1 is false, and let $G = AB = AK = BK$ be a counterexample of minimal order. Then every proper epimorphic image of G is an $\mathfrak{F}$-group. If G has two different minimal normal subgroups N and M, then $G \simeq G/(N \cap M)$ is an $\mathfrak{F}$-group. This contradiction shows that G has exactly one minimal normal subgroup M.

Since M is contained in the nilpotent normal subgroup K of G, M is abelian. The Frattini subgroup $\mathrm{Frat}\, G$ is trivial, since $\mathfrak{F}$ is saturated. Hence M has a complement L in G, so that $G = ML$ and $M \cap L = 1$. Assume that $M \subset C_G(M)$. then $C_G(M) \cap L$ is normal in $ML = G$ and hence must be trivial. Then

$|C_G(M)L| > |ML| = |G|$.

This contradiction shows that $M = C_G(M)$. Then also $M = K$ since K is nilpotent.

Let U be a maximal subgroup of G containing A. Then $U = U \cap AK = A(K \cap U)$, where $K \cap U$ is a proper G-invariant subgroup of K. Therefore $K \cap U = 1$ and $A = U$ is a maximal subgroup of G. Similarly B is a maximal subgroup of G. - Since K is contained in every non-trivial normal subgroup of G, it follows that A and B are $\mathfrak{F}$-maximal in every epimorphic image of G, so that they are $\mathfrak{F}$-projectors of G. But it is well-known that the $\mathfrak{F}$-projectors of a finite group are conjugate (see P Schmid [22]), so that $A = B = G$ is an $\mathfrak{F}$-group. This contradiction proves the theorem.

Corollary 2.2. *Let the finite group* $G = AB = AK = BK$ *be the product of three subgroups* A, B *and* K *where* K *is nilpotent and normal in* G. *If* A *and* B *are nilpotent (supersoluble), then* G *is nilpotent (supersoluble).*

2.3 Remark. There exist finite groups which are not supersoluble, but all their proper subgroups are supersoluble and whose orders contain exactly three primes. These have a triple factorization $G = AB = AK = BK$ with three supersoluble subgroups A, B and K, where K normal in G. This shows that in 2.1 and 2.2 the normal subgroup K has to be nilpotent.

2.4 Problem. Does Theorem 2.1 still hold when the subgroup K of

$$G = AB = AK = BK$$

is no longer normal in G? (Kegel has shown in [16] that Corollary 2.2 holds when the subgroup K is not normal in G.)

3. Locally finite groups

A locally finite group $G = AB = AK = BK$ which is the product of three abelian subgroups A, B and K, where K is normal in G, need not be nilpotent, as the following example shows.

3.1 Example. For each odd prime p let G_p be a metacyclic p-group of class $\geq p$ which has a triple factorization

$$G_p = A_pB_p = A_pK_p = B_pK_p,$$

where A_p, B_p and K_p are cyclic and K_p is normal in G_p (see [24, Example 1]). The direct product $G = \mathrm{Dr}_p\, G_p$ can be written as

$$G = AB = AK = BK,$$

where $A = Dr_p A_p$, $B = Dr_p B_p$ and $K = Dr_p K_p$. It is easy to see that G satisfies the required conditions, is not nilpotent and has Prüfer rank 2.

3.2 Remark. In [15] Holt and Howlett give an example of a metabelian p-group G = AB = AK = BK where A and B are elementary abelian subgroups of G and K is an abelian normal subgroup of G; G is locally nilpotent, but not hypercentral.

The following two results are proved in Amberg [2].

Theorem 3.3. *Let the locally finite group* G = AB = AK = BK *be the product of two hypercentral subgroups* A *and* B *and a locally nilpotent normal subgroup* K *of* G. *Then* G *is locally nilpotent.*

The proof of this result relies heavily on the fact that the hypercentral subgroups $A \neq 1$ and $B \neq 1$ have non-trivial centres.

Theorem 3.4. *Let the locally finite group* G = AB = AK = BK *be the product of three locally nilpotent subgroups* A, B *and* K, *where* K *is normal in* G. *If for every prime* p *the maximal* p-*subgroups of* G *are conjugate, then* G *is locally nilpotent.*

3.5 Problem. Let the locally finite group G = AB = AK = BK be the product of three subgroups A, B and K, where K is locally nilpotent and normal in G. If A and B are locally nilpotent (locally supersoluble), under which conditions is G then locally nilpotent (locally supersoluble)?

To mention some further positive answers to these questions we recall the definition of a class of locally finite-soluble groups that was introduced by Gardiner, Hartley and Tomkinson in [13].

Let $\mathfrak{U}$ be the largest subgroup closed class of locally finite groups with the following properties:

(i) Every $\mathfrak{U}$-group has a finite (subnormal) series with locally nilpotent factors,

(ii) For every set of primes π the maximal π-subgroups of G are conjugate.

Now let $\mathfrak{K}$ be a subgroup and image closed subclass of $\mathfrak{U}$.

A $\mathfrak{K}$-formation $\mathfrak{F}$ is a class of $\mathfrak{K}$-groups which is closed under the forming of epimorphic images and which is *residual with respect to* $\mathfrak{K}$, i.e. if

$$G \in \mathfrak{K},\ G/N_\alpha \in \mathfrak{F} \text{ for every } \alpha \text{ in the index set I, then } G/\bigcap_\alpha N_\alpha \in \mathfrak{F}.$$

In the paper of Gardiner, Hartley and Tomkinson, *saturated* $\mathfrak{K}$-formations are then defined which coincide for finite groups with the usual definitions. The following generalization of a theorem of Carter and Hawkes can be proved:

Let $\mathfrak{F}$ be a saturated $\mathfrak{K}$-formation. If the $\mathfrak{F}$-residual D *of the $\mathfrak{K}$-group* G *is abelian, then* D *has complements in* G *and all these complements are conjugate in* G.

Using this Amberg and Halbritter have proved the following result (unpublished).

Theorem 3.6. *Let $\mathfrak{F}$ be a saturated $\mathfrak{K}$-formation. Let the $\mathfrak{K}$-group*

$$G = AB = AK = BK$$

be the product of three subgroups A, B *and* K, *where* K *is nilpotent and normal in* G. *If* A *and* B *are $\mathfrak{F}$-subgroups, then* G *is an $\mathfrak{F}$-group.*

The proof of this theorem is by induction on the nilpotency class of K, but it seems likely that the result also holds when K is only locally nilpotent.

4. Mixed groups

A soluble group G is a *minimax group* if it has a finite series whose factors are finite or infinite cyclic or quasicyclic of type p^∞; the number of infinite cyclic factors in such a series is the *torsion-free rank* $r_0(G)$ of G, the number of factors of type p^∞ for the prime p is the *p^∞-rank* $m_p(G)$ of G and the *minimax rank* of G is

$$m(G) = r_0(G) + \sum_p m_p(G).$$

The following general theorem contains results that have been provided in a number of papers by Amberg, Franciosi and de Giovanni (see [5], [6], [7], [9]; see also Halbritter [14]).

Theorem 4.1. *Let the group* G = AB = AK = BK *be the product of two subgroups* A *and* B *and a normal minimax subgroup* K *of* G.

(a) *If* A, B *and* K *satisfy some nilpotency requirement $\mathfrak{N}$, then* G *satisfies the same nilpotency requirement $\mathfrak{N}$. Here $\mathfrak{N}$ can for instance be chosen to be any of the following group classes:*

(i) *the classes of nilpotent, hypercentral or locally nilpotent groups,*

(ii) *the classes of* FC-*nilpotent,* FC-*hypercentral groups and locally* FC-*nilpotent groups,*

(iii) *the classes of nilpotent-by-$\mathfrak{X}$, hypercentral-by-$\mathfrak{X}$ and (locally nilpotent)-by-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is the class of finite, periodic, polycyclic, Chernikov or minimax groups,*

(iv) *the class of finite-by-nilpotent groups,*

(v) *the class of locally polycyclic groups.*

(b) *Let* K *be hypercentral. If* A *and* B *are supersoluble (hypercyclic resp. locally supersoluble), then* G *is supersoluble (hypercyclic resp. locally supersoluble).*

Remarks on the proof of Theorem 4.1. (a) In the proof of this general theorem each of the statements has to be checked individually. The proofs are by way of contradiction, inducting on the minimax rank m(K).

(b) The case when K is nilpotent can be reduced to the case when K is abelian by using so called *'theorems of Hall's type'*:

If K *is a nilpotent normal subgroup of the group* G *such that* G/K' *is an* $\mathfrak{N}$*-group, then* G *is an* $\mathfrak{N}$*-group.*

For the class $\mathfrak{N}$ of nilpotent groups this is a well-known result of P Hall (see [19, Part 1, Theorem 2.27, p.56]). For most of the other group classes $\mathfrak{N}$ in Theorem 4.1 the above statement can also be proved.

(c) When K is a torsion-free abelian minimax group, some facts about the automorphism group Aut K are used. In particular a theorem of Baer depending on Dirichlet's Unit theorem of Algebraic Number Theory says that in this case *soluble subgroups of* Aut K *are minimax groups.*

(d) Essential use is also made of cohomological results such as the following theorem of D Robinson in [20]. As usual, $H^i(Q,M)$ denotes the i-th cohomology group of the group Q with coefficients in the Q-module M.

Let Q *be a locally nilpotent group and* M *a* Q*-module such that* $Q/C_Q(M)$ *is hypercentral and* $H^0(Q,M) = 0$. *If* M *is an artinian* Q*-module, then* $H^n(Q,M) = 0$ *for every non-negative integer* n.

These cohomological results can be used to show that the two supplements A and B of K in G are in fact complements and that they are conjugate which easily leads to a contradiction as in the proof of Theorem 2.1.

Three lemmas

The proof of the first statement of Theorem 4.1 that G is nilpotent whenever A, B and K are nilpotent, is much simpler and does not require cohomological results or facts depending on algebraic number theory. The following *'extension lemma'* gives a useful criterion for a group with a nilpotent triple factorization to be nilpotent (see Robinson [21]).

Lemma 4.2. *Let the group* G = AB = AK = BK *be the product of three nilpotent subgroups* A, B *and* K, *where* K *is normal in* G, *and assume that the Baer radical of* G *is nilpotent. If there exists a normal subgroup* N *of* G *such that the factorizer* X(N) *of* N *in* G *and the factor group* G/N *are nilpotent, then* G *is nilpotent.*

Proof. Clearly K is contained in the Baer radical R of G. Now X(N) is subnormal in G since it contains N; therefore $X(N) \subseteq R$. Since $G = AK$,

$$[A \cap BN, {}_rG] \subseteq [A \cap BN, {}_rA]\, R' = R'$$

for a sufficiently large integer r. Therefore $(A \cap BN)R'/R'$ is contained in some term of the upper central series of G/R' with finite ordinal type. Of course a similar statement is true of $(B \cap AN)R'/R'$, so it follows that NR'/R' is contained in some term with finite ordinal type of the upper central series of G/R'. Consequently G/R' is nilpotent. By hypothesis R is nilpotent and a well-known theorem of P Hall shows that G is nilpotent.

To apply the 'extension lemma' we need to know that the Baer radical of G is nilpotent. This is ensured by the next lemma.

Lemma 4.3. *Let* N *be a normal subgroup with finite Prüfer rank* r *of the Baer group* G.

(a) *If* N *is a radicable abelian* p*-group, then* N *is contained in the* r*-th term* $Z_r(G)$ *of the upper central series of* G.

(b) *If the torsion subgroup of* N *is a Chernikov group and the factor group* G/N *is nilpotent, then* G *is nilpotent.*

Proof. (a) If H is a finitely generated subgroup of G, then H is a nilpotent subnormal subgroup of G. Write

$$N_i = [N, \underbrace{H, \ldots, H}_{i}]$$

for every positive integer i. Then $N_t = 1$ for some t. Since every N_i is radicable, it follows that N_i is a direct factor of N_{i-1} for all $i \le t$. Hence, $t \le r$ and thus $N_r = 1$. Therefore also $[N, \underbrace{G, \ldots, G}_{r}] = 1$ and so $N \subseteq Z_r(G)$.

(b) Since the torsion subgroup T of N is a Chernikov group, its finite residual J is a radicable abelian torsion group with finite Prüfer rank, and T/J is finite. Clearly N/T is a torsion-free nilpotent normal subgroup of G/T with finite Prüfer rank and so $N/T \subseteq Z_s(G/T)$ for some positive integer s (see [17, Part 2, p.35]) and G/T is nilpotent. Thus G/J is finite-by-nilpotent and hence nilpotent. By (a) we have that $J \subseteq Z_r(G)$, so that G is nilpotent.

The last lemma is elementary, but very useful (see Amberg [4]).

Lemma 4.4. *Let the group* $G = AB = AK = BK$ *be the product of two subgroups* A *and* B *and an abelian normal subgroup* $K \neq 1$ *of* G *with*

$$A \cap K = B \cap K = 1.$$

If $a = bx \in Z(A)$ *with* $b \in B$ *and* $x \in K$, *and if* $N \neq 1$ *is a normal subgroup of* G *contained in* K *and containing* x, *then* [N,a] *is a normal subgroup of* G *with* $[N,a] \subset N$.

Proof. Clearly Z(A)K = Z(B)K and hence $b \in Z(B)$. Since N is an abelian normal subgroup of G, [N,a] is the set of all [n,a] with $n \in N$. Then [N,a] is a normal subgroup of G = AK, as $a \in Z(A)$.

If $a \in Z(B)$, then $a \in Z(G)$ and therefore $[N,a] = 1 \subset N$. Hence we may assume that $a \notin Z(B)$, so that $x \neq 1$. Assume that N = [N,a]. There exists an element $n = a^*b^* \neq 1$ in N with $1 \neq x = [n,a]$ with $a^* \in A$ and $b^* \in B$. It follows that

$$[a^*b^*,a] = b^{*-1}a^{*-1}a^{-1}a^*b^*a = b^{*-1}a^{-1}b^*a$$

and therefore

$$a = bx = bb^{*-1}a^{-1}b^*a$$

and

$$b^*b^{-1}b^{*-1} = a^{-1} \in B.$$

Since $a \in Z(A)$ and $b \in Z(B)$ it follows that $a = b \in Z(B)$. This contradiction proves the lemma.

Proof of the 'nilpotent statement' of Theorem 4.1.

Let G = AB = AK = BK, *where* A, B *and* K *are nilpotent subgroups of* G *and* K *is a minimax group. Then* G *is nilpotent.*

Assume that this statement is false, and choose among the counterexamples with K of minimal minimax rank a group G for which the sum of the nilpotency classes of A and B is also minimal. By a well-known result of P Hall we may assume that K is abelian (see [19, Part 1, p.56]). If G is finite-by-nilpotent, $|G:Z_n(G)|$ is finite for some non-negative integer n by another theorem of P Hall (see [19, Part 1, p.117]). Since the nilpotent statement holds for finite groups by Theorem 2.1 or Corollary 2.2 it follows that G is nilpotent. This contradiction shows that G is not finite-by-nilpotent.

(i) The case: K is a torsion group

In this case K is a Chernikov group. There exists a finite G-invariant subgroup E of K such that K/E is radicable. If G/E is nilpotent, then G is finite-by-nilpotent, a contradiction. Therefore we may assume that K is radicable.

Let H be an infinite G-invariant subgroup of K. If $H \subset K$, then the factor group G/H and the factorizer X(H) of H in G are nilpotent. By Lemma 4.3(b) the Baer radical of G is nilpotent, so that G is nilpotent by Lemma 4.2. This contradiction shows that every proper G-invariant subgroup of K is finite.

Clearly the normal subgroups $A \cap K$ and $B \cap K$ of G are properly contained in K, so that $C = (A \cap K)(B \cap K)$ is a finite normal subgroup of G and so G/C is not nilpotent. Therefore we may assume that $A \cap K = B \cap K = 1$. For every a in Z(A), the group [K,a] is a normal subgroup of G which is properly contained in K; see Lemma 4.4. Since K is radicable, we have that $[K,a] = 1$. This shows that $Z(A) \subseteq Z(G)$, so that G/Z(G) is nilpotent. This contradiction proves that G is nilpotent when K is a torsion group.

(ii) The general case

The factorizer X(T) of the torsion subgroup T of K is nilpotent by case (i). By Lemma 4.3(b) the Baer radical of G is nilpotent, so that G/T is not nilpotent by Lemma 4.2. Hence we may assume that K is torsion-free.

Then K/K^p is finite for every prime p, so that by case (i) G/K^p is nilpotent. It follows that $[K, {}_rG] \subseteq K^p$ where r is the rank of K. Since K is a minimax group we have $\bigcap_p K^p = 1$. Consequently $[K, {}_rG] = 1$. Hence G is nilpotent and this contradiction proves the nilpotent statement of Theorem 4.1.

5. An Application

The 'nilpotent statement' of Theorem 4.1 has the following consequence, which generalizes results in [1] for polycyclic groups and of Pennington in [17] for finite groups.

Theorem 5.1. *If the soluble minimax group* $G = AB$ *is the product of two nilpotent subgroups* A *and* B, *then each term of the ascending Fitting series of* G *is factorized; in particular the Fitting subgroup of* G *is factorized.*

Proof. Consider the ascending Fitting series of G

$$1 = F_0 \leq F_1 \leq F_2 \leq \dots F_n = G,$$

where F_i is normal in F_{i+1} and F_{i+1}/F_i is the Fitting subgroup of G/F_i; in particular $F_1 = \mathrm{Fitt}(G)$. If X_i is the factorizer of F_i in G for $i \leq n$, then the