CHAPTER I.

KUMMER'S CONFIGURATION.

§ 1. DESMIC TETRAHEDRA.

The eight corners of a cube form a very simple configuration; yet by joining *alternate* corners by the diagonals of the faces we get two tetrahedra such that each edge of one meets two opposite edges of the other, and the figure possesses all the projective features of the most general pair of tetrahedra having this property.

Take an arbitrary tetrahedron of reference $XYZT$, and any point S whose homogeneous coordinates are x, y, z, t. Draw three lines through this point to meet the pairs of opposite edges, and on each line take the harmonic conjugate of S with respect to the intercept between the edges; in this way three new points P, Q, R are obtained, making in all the set of four

$$\begin{array}{ll} P, & (\ \ x, -y, -z, t\ ), \\ Q, & (-x,\ \ y, -z, t\ ), \\ R, & (-x, -y,\ \ z, t\ ), \\ S, & (\ \ x,\ \ y,\ \ z, t\ ). \end{array}$$

Then $PQRS$ and $XYZT$ are a pair of tetrahedra possessing the above property, for PS and QR meet both XT and YZ, and so on; they are the most general pair, for the preceding harmonic construction is deduced from the fact that, by hypothesis, any face of one tetrahedron cuts the other in a complete quadrilateral whose diagonals are the edges in that face. When one tetrahedron is given the other is determined by any one of its corners. Tetrahedra so related are said to be *desmic* and to belong to a *desmic system*.

These two tetrahedra possess further the remarkable property of being in *fourfold perspective*; for the lines PT, QZ, RY, SX are concurrent in $(-x, y, z, t)$, and so on. Thus there are four centres of perspective

$$\begin{array}{llllll} P', & (-x, & y, & z, & t &), \\ Q', & (\ x, & -y, & z, & t &), \\ R', & (\ x, & y, & -z, & t &), \\ S', & (\ x, & y, & z, & -t &), \end{array}$$

and we see that the points P', Q', R' are obtained from S' in the same way as P, Q, R were obtained from S, that is, by changing the signs of two coordinates; hence the tetrahedra $P'Q'R'S'$ and $XYZT$ belong to the same desmic system.

It has just been shown that the twelve corners of the three tetrahedra $PQRS$, $P'Q'R'S'$, $XYZT$ lie by threes on sixteen lines, and from this it follows that each pair of tetrahedra is in fourfold perspective. Now the property of being in fourfold perspective is an equally good definition of desmic tetrahedra and all the other properties can be deduced from this. We are thus led to include *three* tetrahedra in every desmic system.

None of the projective features of the figure are lost by taking X, Y, Z to be the infinite ends of a frame of rectangular axes

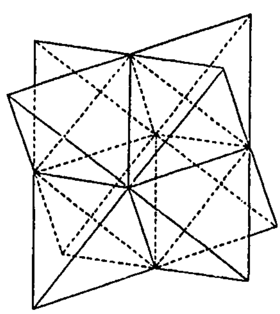

FIG. 1.

meeting in an origin T, and by supposing that $x = y = z$; then P, Q, R become the *images* of S in the axes, and P', Q', R', S' become the images of S in the planes of reference and in the origin respectively. We have, in fact, the corners of a cube, and the diagonals of the faces are the edges of the regular tetrahedra $PQRS$ and $P'Q'R'S'$. The figure is one that is easily conceived, and its desmic properties are readily discerned by geometrical intuition. Sets of parallel edges of the cube show the tetrahedra in three perspective aspects, and the diagonals show that the centre is also a centre of perspective. Of the intersections of the edges, six are the centres of the faces of the cube and are the corners of a regular octahedron. The remaining six are at infinity.

It is interesting to notice that these twelve points of intersection are the corners of another desmic system of tetrahedra formed with the same eighteen edges; in the figure the three new tetrahedra are infinite and wedge-shaped, each being formed by two opposite faces of the cube and the planes containing their parallel diagonals.

Since the figure is defined by intersecting lines, it is self-reciprocal, and to every point-theorem there is a corresponding plane-theorem; in particular, the faces of any two tetrahedra can be paired in four ways so that the lines of intersection lie in a face of the third tetrahedron. The geometrical properties of the figure are deducible from the identity

$$(a-b-c+d)(-a+b-c+d)(-a-b+c+d)(a+b+c+d) + (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) \equiv 16abcd,$$

in which the letters may be regarded as current coordinates of either a plane or a point.

The two following results may be taken as examples.

The twelve centres of similitude of four spheres are the corners of a desmic system.

If three tetrahedra belong by pairs to different desmic systems, the remaining tetrahedra of the three systems belong to another system.

Desmic systems were first investigated by Stephanos* and are so named because the three tetrahedra belonging to one system are members of a pencil (faisceau, δεσμός) of quartic surfaces. The general surface of the pencil has twelve nodes and is the subject of a memoir by Humbert†; its equation may be written in the symmetrical form

$$\lambda(x^2t^2+y^2z^2)+\mu(y^2t^2+z^2x^2)+\nu(z^2t^2+x^2y^2)=0,$$

where

$$\lambda+\mu+\nu=0,$$

and the three tetrahedra are

$$(x^2-t^2)(y^2-z^2)=0, \quad (y^2-t^2)(z^2-x^2)=0, \quad (z^2-t^2)(x^2-y^2)=0,$$

the sum of the left sides being identically zero. For further details and references see a paper by Schroeter‡; an application to Spherical Trigonometry is given by Study§.

* *Darboux Bulletin* (1879), sér. 2, III, 424.
† *Liouville* (1891), sér. 4, VII, 353.
‡ *Crelle* (1892), CIX, 341.
§ *Mathematical Papers*, Chicago Congress (1893), p. 382.

§ 2. THE GROUP OF REFLEXIONS.

A *group of operations* is a set of operations such that the resultant of any number taken in any order is an operation of the set*; in particular the repetition of a single operation any number of times is equivalent to some member of the group.

The fact that by successive reflexions in the axes only a finite number of points are obtained from one arbitrary point shows that the operations of reflecting belong to a group. Considered algebraically the operations consist in changing the signs of two of the coordinates. Let the symbol A denote the operation of changing the signs of y, z, and therefore of changing S into P. The repetition of the operation A changes P back into S again, and this is expressed by the symbolic equation

$$A^2 = 1.$$

Here 1 denotes the *identical operation*, which does not alter the position of the point to which it is applied; we infer that it must be included in every group to which A belongs. Similarly let the operation B change the signs of z, x, and C those of x, y. Then $B^2 = 1$ and $C^2 = 1$. Further, if B and C are performed successively in either order the result is the same as when y and z, but not x, are changed in sign; this is expressed by the symbolic equations

$$BC = CB = A.$$

In other words, B and C are *permutable* and their *product* is A. These equations, with others deduced by symmetry, are sufficient to show that the four operations

$$1,\ A,\ B,\ C$$

form a *group*, for any combination of them can be reduced to one of themselves. The *multiplication table*

	A	B	C
A	1	C	B
B	C	1	A
C	B	A	1

is a convenient way of representing the equations

$$A^2 = B^2 = C^2 = 1,$$

$$BC = CB = A,\quad CA = AC = B,\quad AB = BA = C.$$

* For a complete definition and fuller explanations see Burnside's *Theory of Groups*, Chap. II.

It may happen that some of the members of a group form a group by themselves. In this case the smaller group is called a *subgroup* of the larger. For example $(1, A)$ is a subgroup of $(1, A, B, C)$; $(1, B)$ and $(1, C)$ are also subgroups.

When two permutable groups are given, a third group can be obtained by combining the members of one with the members of the other in all possible ways, and is called the *product* of the first two groups. For example, the group of reflexions is the product of any two of its subgroups. The *order* of the product group, that is, the number of its members, is the product of the orders of the first two groups.

§ 3. THE 16_6 CONFIGURATION.

In space of three dimensions a point or a plane may be represented by four symbols $\alpha, \beta, \gamma, \delta$ used homogeneously. The condition of *incidence* of two elements $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ of different kinds may be taken to be

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta' = 0.$$

On account of the perfect reciprocity between point and plane in projective geometry, every theorem that will be proved has its correlative theorem: it will not be necessary to state the second result in every case.

It is immediately verifiable that the plane

$$(\alpha, \quad \beta, \quad \gamma, \quad \delta)$$

contains the six points

$$\begin{array}{l} (\delta, \ -\gamma, \ \beta, \ -\alpha), \\ (\delta, \ \gamma, \ -\beta, \ -\alpha), \\ (\gamma, \ \delta, \ -\alpha, \ -\beta), \\ (-\gamma, \ \delta, \ \alpha, \ -\beta), \\ (-\beta, \ \alpha, \ \delta, \ -\gamma), \\ (\beta, \ -\alpha, \ \delta, \ -\gamma), \end{array}$$

and this is the foundation of the configuration; the preceding six incidences are true for all values of the symbols, and we may therefore substitute the members of any of the last six rows for $\alpha, \beta, \gamma, \delta$ respectively and so obtain other sets of incidences. It will be found that only sixteen different points and sixteen different planes can be obtained in this way, and this is due to the fact that the operations of permuting and sign-changing involved in these substitutions belong to a *group*. In order to explain the formation of this group we must introduce symbols for its members.

§ 4. THE GROUP OF SIXTEEN OPERATIONS.

Let A denote the operation of interchanging α, δ and at the same time β, γ, each letter carrying its sign with it; similarly B interchanges β with δ and γ with α, and C interchanges γ with δ and α with β. Further let A' denote the operation of changing the signs of β, γ, B' those of γ, α, and C' those of α, β. Then A', B', C' belong to the group of reflexions which has already been considered, and it is easily seen that A, B, C satisfy symbolic equations of precisely the same form; in other words $(1, A, B, C)$ and $(1, A', B', C')$ are two groups having similar multiplication tables. Since change of order is independent of change of sign, all the members of one group are permutable with those of the other, for example $AB' = B'A$, and consequently the groups themselves are said to be permutable.

By combining the members of these two permutable groups in all possible ways we obtain a set of sixteen operations which evidently form a *group* containing $(1, A, B, C)$ and $(1, A', B', C')$ as subgroups. All the sixteen planes of the configuration are obtained by operating on any one of them, say $(\alpha, \beta, \gamma, \delta)$, with the members of the group, and the six points lying in each plane are obtained by operating on the set given in § 3 with the corresponding member of the group; for, the condition of incidence is unaffected when the same operation is performed on point and plane. We may clearly use the symbol of operation to denote the point or plane obtained from $(\alpha, \beta, \gamma, \delta)$ by that operation, thus (1) denotes $(\alpha, \beta, \gamma, \delta)$ and (AB') denotes $(\delta, -\gamma, \beta, -\alpha)$, and so on; we have seen that the plane (1) or $(\alpha, \beta, \gamma, \delta)$ contains the points

$$(AB'),\ (AC'),\ (BC'),\ (BA'),\ (CA'),\ (CB'),$$

and we deduce that the plane (A), or $(\delta, \gamma, \beta, \alpha)$, contains the points

$$(B'),\ (C'),\ (CC'),\ (CA'),\ (BA'),\ (BB'),$$

and so on.

The group of sixteen operations, which will be referred to simply as *the group*, contains many subgroups. Any two operations and their product form, with the identical operation, a subgroup: two examples are $(1, AB', BC', CA')$ and $(1, AC', BA', CB')$. Further the group can be arranged in many ways as the product of two subgroups; one arrangement arises from the definition and another from the two preceding subgroups. These are shown by the multiplication tables

1	A'	B'	C'
A	AA'	AB'	AC'
B	BA'	BB'	BC'
C	CA'	CB'	CC'

1	AC'	BA'	CB'
AB'	A'	CC'	B
BC'	C	B'	AA'
CA'	BB'	A	C'

It is an easy exercise to verify the following table:

order of subgroup = 2, 4, 8,
number of subgroups = 15, 35, 15.

§ 5. THE INCIDENCE DIAGRAM.

We have thus found the coordinates of sixteen points and sixteen planes such that six points lie in each plane and six planes pass through each point. The most general 16_6 configuration, which is defined by these properties, can be reduced to the preceding form by a proper choice of coordinates.

The whole scheme can be exhibited very compactly by the following artifice. Since the subgroups $(1, A, B, C)$ and $(1, A', B', C')$ obey the same laws, they may be represented by the same symbols: the members of the two subgroups will be distinguished by the position they occupy in a compound symbol. Every member of the product group will be represented by a two-letter symbol in which the first letter will represent a member of $(1, A, B, C)$ and the second a member of $(1, A', B', C')$. The operations of each of these subgroups will be denoted by

$$d,\ a,\ b,\ c,$$

so that in either position d represents the identical operation. The multiplication table

	d	a	b	c
d	d	a	b	c
a	a	d	c	b
b	b	c	d	a
c	c	b	a	d

which is fundamental for this representation, applies to both the first and the second letters in a compound symbol, and the table showing the product of the two subgroups $(1, AB', BC', CA')$ and $(1, AC', BA', CB')$ takes the form

dd	*ac*	*ba*	*cb*
ab	*da*	*cc*	*bd*
bc	*cd*	*db*	*aa*
ca	*bb*	*ad*	*dc*

The sixteen symbols, which in the first instance denote operations, can, as is explained above, be used to denote both points and planes; it will be found that no confusion arises from this, but that, on the contrary, the duality of the configuration is clearly brought out by this nomenclature. Now any row is obtained from any other row by one of the operations *ab*, *bc*, *ca*, and any column from any other column by one of the operations *ac*, *ba*, *cb*. Since the plane (*dd*) contains the points represented by the other symbols in the same row and column, it follows that the *six elements incident with any element are given by the row and column containing that element.*

This property of the table is not lost if the rows are permuted in any manner, and also the columns. This, as well as the group property, shows that all the elements are of equal importance in the configuration, although the notation isolates (*dd*), or $(\alpha, \beta, \gamma, \delta)$. To bring this out more clearly the symbols in the table will frequently be replaced by dots, and then we shall have an *incidence diagram* which will be of great use for indicating

·	o	·	·
o	×	o	o
·	o	·	·
·	o	·	·

·	·	·	·
·	×	·	o
·	o	·	×
·	·	·	·

at a glance relations among the elements of the configuration. Thus, for example, in the first diagram the plane × contains the six points o, and the second diagram shows that any two planes have two points in common.

§ 6. LINEAR CONSTRUCTION FROM SIX ARBITRARY PLANES.

By means of the incidence diagram it is easy to prove that the 16_6 configuration can be linearly constructed from six arbitrary planes, and also, reciprocally, from six arbitrary points. It is convenient to use two diagrams, one for planes and the other for points; each is an incidence diagram for the elements contained in it, and two elements, one from each diagram, are incident if they lie on corresponding rows, or columns, but not both.

×	×	×	·
×	·	·	·
·	×	·	·
·	·	×	·

o	o	o	o
·	o	o	·
o	·	o	·
o	o	·	·

Let the first diagram represent any six planes; the positions of the crosses make no suppositions as to the linear dependence of the planes, for the diagram does not indicate that more than three planes pass through the same point. It is required to fill in the remaining ten places, if possible, so as to complete the incidence diagram and obtain a 16_6 configuration. The noughts in the second diagram represent ten of the twenty points of concurrence of the six planes, taken by threes: for example, the three crosses in the first row determine the last nought of the first row. Now every row and column in the second diagram, taken together, contain enough noughts to determine a plane of the configuration; in this way the remaining ten planes are found and the first diagram may be completed.

Hence a 16_6 configuration can be constructed from six arbitrary planes in at least one way, and therefore involves eighteen arbitrary constants. Now the system considered in § 3 contains the three ratios $\alpha : \beta : \gamma : \delta$ and fifteen constants implied in the choice of a particular set of homogeneous coordinates. We infer that the general configuration can be represented in this way.

We shall now investigate the preceding process of constructing the configuration in greater detail, and prove that six given planes determine *twelve* configurations.

Let five planes in general position be denoted by 1, 2, 3, 4, 5, their lines of intersection by two-figure symbols and their common points by three-figure symbols. There are twelve different cyclical arrangements of the planes and each gives a skew pentagon formed by the intersections of the planes taken in order. Thus corresponding to the arrangement

1 2 3 4 5 1

there is a pentagon with sides

12 23 34 45 51

and corners

123 234 345 451 512 ....P.

Each side contains, besides two corners of the pentagon, one other point of the system, where it meets the *opposite* plane, making the set

124 235 341 452 513 ....*Q*,

and these, when arranged in the order

135 352 524 241 413..................*P′*,

are the corners of the pentagon corresponding to the cyclical arrangement of planes 13524. The relation between the pentagons *P* and *P′* is mutual, and so the twelve pentagons can be divided into six pairs, the members of each pair being mutually inscribed and circumscribed.

We next prove that the pentagons whose corners are *P* and *Q*, taken in the order given, are so related that when they are projected from any point on to any plane, five intersections of pairs of sides are collinear. Giving the projections the same names as the points and lines in space, we see that the sets of points

341 123 513

and 512 452 235

are collinear, lying on 13 and 25 respectively. Therefore, by Pascal's theorem, the intersections of

(513, 512), or 51, and (235, 341)

(512, 123), or 12, and (341, 452)

(123, 235), or 23, and (452, 513)

are collinear, and similarly for the other pairs of sides. Hence the theorem is proved.

Two skew pentagons, which are so related that the five lines from any point to meet pairs of corresponding sides are coplanar, are said to be in *lineal position*; we have now proved that the twelve pentagons formed by the intersections of five planes taken in different orders can be arranged in six pairs such that if the corners of one pentagon are taken alternately a new pentagon is formed which is in lineal position with the other member of the pair. In this way we get twelve pairs of pentagons in lineal position.

Conversely, instead of projecting from an arbitrary point, take any sixth plane 6; its intersections with the lines 12 and (134, 245), 23 and (245, 351) determine two lines meeting in a point which must be collinear with the points where 6 cuts 34