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THE DISTRIBUTION OF PRIME NUMBERS

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FOREWORD

As an introduction to the distribution of primes Ingham's tract is still unsurpassed, combining an economy of detail with a clarity of exposition which eases the novice's way into this, sometimes technically ferocious, area. In spite of the fact that it has been out of print for many years I usually place it at, or near, the top of the reading list for graduate students.

When Ingham wrote his tract, the theory of the distribution of primes depended almost exclusively on the theory of the Riemann zeta-function. To a large extent this is still true, but there have been a number of important developments which depend, at least in part, on various aspects of sieve theory or the use of exponential sums of various kinds.

In 1934 the theory of the distribution of primes in an arithmetic progression $qn+a$ with $(q, a)=1$ was in all essentials the precise analogue of the case $q=1$ and so was ignored by Ingham. In the intervening years there has been significant progress concerned particularly with the situation when the modulus q is large, and this means that a modern exposition of the theory of primes has to cover a good deal more ground. For a good introduction to this material see Davenport (1980).

Ingham was very careful to connect his exposition with the original research which led to the theory he describes. There have been many refinements and developments of this theory in the intervening years and below we indicate some of them as they relate to Ingham's text. Ingham's brief was to leave the deeper properties of the Riemann zeta-function for exposition by Titchmarsh in a parallel tract, Titchmarsh (1930), and much of the work on the distribution of primes over the last 50 years is intimately related to such properties of the zeta-function.

Introduction

§ 5. There have been a number of attempts to obtain more precise estimates for $\pi(x)$ by methods related to those of Chebyshev, and a good deal is now known about them and their limitations. See Nair (1982).

§ 6. Edwards (1974) has devoted a book to a study of Riemann's seminal memoir and all later work which stems from it, and is perhaps now the most accessible source for much of the early material.

§ 7. Landau's function theoretic method mentioned here has been central to the improved results described in the comments on Chapter III, § 13 below.

§ 8. Selberg (1949) discovered a 'real variable', or 'elementary' proof of the prime number theorem. Thus the view expressed in § 8 is nugatory. Selberg first establishes the approximate identity

$$\psi(x)(\log x)+\sum_{n\leq x}\Lambda(n)\psi(x/n)=2x(\log x)+O(x),$$

or rather an approximate identity which is essentially equivalent to this. That is relatively easy. Harder, and harder to motivate, is the Tauberian process of extracting asymptotic information about $\psi(x)$. Several of the numerous expositions are unsatisfactory in this regard. There is a variant of the proof in Erdős and Selberg (1949), and of the expositions Diamond (1982) and Hardy and Wright (1979, Chapter XXII) are reliable without getting bogged down in the details.

A number of authors have adapted Selberg's method to obtain quite a good error term in the prime number theorem. The best estimate so far obtained by this method is

$$O(x\exp(-\log^a x))$$

with a any number smaller than 1/6. This is due to Lavrik and Sobirov (1973) and should be compared with Theorem 23 and the improved result stated below.

For a completely different elementary proof, see Daboussi (1984).

§ 9. The truth, or otherwise, of the Riemann hypothesis has still not been established, despite quite frequent claims which are often sensationalised by the media.

§ 10. Electronic computers have enabled the calculations to be greatly extended. See, for example, Riesel (1985), Chapters 1 and 2. A small correction has been made to the table on page 7. Skewes (1933, 1955) showed that, contrary to everyone's belief

in the 1920s and early 1930s, Littlewood's theorem is not a pure 'existence' theorem, and that the first sign change occurs for some x not exceeding $10_4(3)$ where

$$10_1(x) = 10^x,\ 10_2(x) = 10^{10_1(x)},$$

and so on. See pages 110–12 of Littlewood's delightful *Miscellany*, (Littlewood, 1986) for an account of this work. Later, by a different method, Lehman (1966) reduced Skewe's constant considerably, to $1{\cdot}65 \times 10^{1165}$, and recently te Riele (1986) has shown that between $6{\cdot}62 \times 10^{370}$ and $6{\cdot}69 \times 10^{370}$ there are 10^{180} successive integers x for which $\pi(x) > \mathrm{li}(x)$.

§ 11. Titchmarsh's tract was later expanded into a celebrated text (Titchmarsh, 1986) which is still the standard work on the subject. With regard to upper bounds for $p_{n+1} - p_n$, Ingham himself (1937) made the most startling progress by showing that the bound he states here holds with $\vartheta > 5/8$. Heilbronn (1933) had earlier obtained the lower bound $1 - 1/250$ and more recently Montgomery (1969) (see also Montgomery, 1971), Huxley (1972), Iwaniec and Jutila (1979), Heath-Brown and Iwaniec (1979), Iwaniec and Pintz (1984) and Mozzochi (1986), have obtained 3/5, 7/12, 13/23, 11/20, 23/42 and 1051/1921, respectively. In fact there has been much activity in this area over the last 20 years. For an account of some of this see Chapter 12 of Ivić (1985).

It is now known that $p_{n+1} - p_n$ is sometimes significantly larger than $\log p_n$. Following work of Erdős (1935), Rankin (1938, 1963) and Schönhage (1963), Maier and Pommerance (1990) have shown that

$$\limsup_{n\to\infty} \frac{(p_{n+1} - p_n)(\log\log\log p_n)^2}{(\log p_n)(\log\log p_n)(\log\log\log\log p_n)} \geqslant 4e^{\gamma}/c,$$

where γ denotes Euler's constant and c satisfies $c = 3 + e^{-c}$.

In the opposite direction it is still not known whether

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < \infty.$$

However, Erdős (1940), established that there is a real number $C < 1$ such that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \leqslant C.$$

Bombieri and Davenport (1966) showed that $C = 0{\cdot}467$ is permissible, and after some small improvements by Pil'tai (1972) and Huxley (1973, 1977), Maier (1988) has obtained $C = 0{\cdot}248$.

In much of the work in connection with differences between consecutive primes, sieve theory has played a significant role. The two standard works on sieves are Halberstam and Richert (1974) and Hooley (1976).

Chapter I

§ 3. For the further developments of sieve theory see the texts quoted above. The Goldbach binary problem is still unsolved. The constant k mentioned here has now been shown to satisfy $k \leqslant 19$ (Riesel and Vaughan, 1983), and Vinogradov (1937) has shown that every large odd number is the sum of three primes. Also Montgomery and Vaughan (1975) have shown that there is a positive constant c such that the number of even numbers not exceeding x which are not the sum of two primes is at most $O(x^{1-c})$, and Chen (1973, 1978) has shown that every large even number is the sum of a prime and a number having at most two prime factors.

Chapter II

§ 3. With regard to the reference in the footnote and more recent material see Titchmarsh (1986), Chapters V, VI, VIII, XIV.

§ 4. With regard to the comment on (10) at the end of this section, see the remarks below on § 9 of Chapter III.

§ 11. There are now other proofs of the prime number theorem which require only information about $\zeta(s)$ in the closed half-plane $\sigma \geqslant 1$, due to Wirsing (1967) and Halász (1969). They are important because they have arisen as special cases of solutions to quite general problems concerning mean values of multiplicative functions, and have spawned a large volume of investigations into mean value theorems for general classes of additive and multiplicative functions.

§ 12. In view of Selberg's elementary proof of the prime number theorem (Selberg, 1949) there is no longer any distinction between elementary and transcendental here.

Chapter III

§ 8. A modern reference for footnote 2 on page 58 is Titchmarsh (1986), (2.12.8).

§ 9. With a view to calculating a relatively large value for the constant a, there is an interesting extremal question with regard to non-negative cosine polynomials with non-negative coefficients akin to that mentioned at the end of § 4 of Chapter II. Here one is interested in the extremal value of

$$(\sqrt{c_1}-\sqrt{c_0})^2/(c_0+c_1+\cdots+c_m).$$

This question has not been answered completely. See Kondrat'ev (1977).

§ 11. In view of alternative proofs of the prime number theorem with reasonable error terms, and in particular Selberg's elementary method, one can ask for an inverse of Theorem 22, i.e. what zero-free region for the zeta-function can be deduced from a given error term in the prime number theorem. Turán (1950) discovered that an error term of the form

$$O(x\exp(-a(\log x)^b))$$

implies that $\zeta(s)$ is non-zero for

$$\sigma \geqslant 1-c(\log(2+|t|))^{(b-1)/b}$$

and in this special case this gives the desired converse. In particular this shows that a non-trivial zero-free region can be obtained by an 'elementary' proof.

Turán found many variants of the above theorem, and they are described in his book (Turán, 1984). Much of this work is intimately connected with research on irregularities of distribution, the topic treated by Ingham in Chapter V.

Ultimately, Pintz (1982a, b, 1984) established a converse to Theorem 22 which is valid under very general conditions.

§ 13. The zero-free region has been improved. It is now known that $\zeta(s)$ has no zeros in a domain of the form

$$\sigma > 1-a(\log t)^{-b}(\log\log t)^{-c}$$

with $b<1$. Chudakov (1936) was the first to obtain a region of this kind, with b any real number exceeding 7/8 and $c=0$. The

best value so far obtained for b is 2/3 due to Korobov (1958) and Vinogradov (1958). With regard to the value of c, Korobov and Vinogradov apparently claim more ($c = 0$) than can be established by their methods, namely $c = 1/3$, and no retraction has ever been made. Ingham, in a review (1964) is justifiably scathing.

These improved zero-free regions are a consequence of estimates for exponential sums that follow from a mean value theorem for such sums known as the Vinogradov Mean Value Theorem. The exponential sum estimates imply bounds for the zeta-function in the neighbourhood of the line $\sigma = 1$, and the zero-free region is obtained then by an application of the function theoretic method of Landau mentioned in § 7 of the Introduction.

The zero-free region mentioned above with $b = 2/3$, $c = 1/3$, when applied to Theorem 22, gives at once the best error term known in the prime number theorem, i.e. the error

$$O(x \exp(-(\log x)^{3/5}(\log \log x)^{-1/5})).$$

There are good expositions of this material in Ivić (1985) Chapters 6 and 12 and Richert (1967).

§ 14. For some discussions related to the Riemann hypothesis see Edwards (1974), § 1.9 and the notes to Chapter 1 of Ivić (1985), § 4.F of Ribenboim (1988) and Chapters XIV and XV of Titchmarsh (1986).

Chapter IV

§ 2. For developments on the distribution of zeros see Ivić (1985), § 1.4 and Chapter 11, and Titchmarsh (1986), Chapter 9.

§§ 5 and 6. The explicit formulae established here are the prototypes of many analogous formulae which have been discovered in other, not always closely related, areas. Apart from the generalisations by Weil (1952), and to Dirichlet L-functions, for which excellent sources are Chapter 5 of Patterson (1988) and Chapter 19 of Davenport (1980) respectively, there are, for instance, explicit formulae connected with rational functions over finite fields (see Chapter II of Schmidt (1976)) and with Riemann surfaces (see Selberg (1956), Hejhal (1976)).

§ 8. With regard to the order of the difference $\psi(x)-x$, see the comments below to Chapter V, § 8.

Chapter V

This chapter has become very dated, at least with regard to the proof of Littlewood's theorem, Theorem 34, largely as a consequence of Ingham's own investigations. In Ingham (1936), by considering a suitable weighted average of $\psi(x)-x$, he was led to the weighted partial sum

$$\sum_{0<\gamma\leq T}(1-\gamma/T)\frac{\sin\gamma t}{\gamma}$$

and was able thereby to avoid the use of the Phragmén–Lindelöf Theorem. Later, in the notes to Hardy's collected works (Hardy, 1967, page 99) he briefly indicates a further improvement. The later work of Skewes with regard to the first sign change of $\pi(x)-\mathrm{li}(x)$ mentioned above in the comments to § 10 of the Introduction is based on a variant of the proof given here. As it stands the proof is non-effective as far as the determination of the implicit constant is concerned. However, Skewes avoids this by a small change in the division of cases, in essence Riemann hypothesis 'almost true' and its negation. The later work of Lehman and te Riele is based on Turán's method, mentioned in the comments to § 11 of Chapter III. All work of this kind requires a quantitative theorem on diophantine approximation, such as Dirichlet's theorem, Theorem J, or something equivalent. Unfortunately Dirichlet's theorem is a theorem on homogeneous diophantine approximation whereas the underlying problem in this instance is an inhomogeneous one – ideally one wishes to arrange that, for suitable t, each of the quantities $\sin(\gamma t)$ with $0<\gamma\geqslant T$ are all close to $+1$ or all close to -1. However, the inhomogeneous theory (Kronecker's theorem – see Chapter XXIII of Hardy and Wright (1979)) only gives qualitative estimates. Turán's method overcomes this difficulty and provides quantitative estimates in an inhomogeneous situation. The estimates are relatively weak and do not improve on Littlewood's theorem. However, they are more flexible and more amenable to

calculation. For further reading in this area see Chapter 12 of Ivić (1985) and, especially, Turán (1984) and Pintz (1982a, b, 1984).

Montgomery has suggested that, provided that the Riemann hypothesis holds and linear forms in the imaginary parts of the zeros above the real axis are not abnormally small, then

$$\limsup_{x\to\infty} (\psi(x)-x)x^{-1/2}(\log\log\log x)^{-2} = 1/(2\pi)$$

and

$$\liminf_{x\to\infty} (\psi(x)-x)x^{-1/2}(\log\log\log x)^{-2} = -1/(2\pi).$$

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PREFACE

The subject of this tract is the theory of the distribution of the prime numbers in the series of natural numbers. A chapter on the 'elementary' theory has been included for its historical interest and for the intrinsic interest of the methods employed, but the major part of the book is devoted to the analytical theory founded on the zeta-function of Riemann. The tract is thus a companion to No. 26 of the series, 'The zeta-function of Riemann' by Prof. E. C. Titchmarsh, published in 1930, but the logical sequence of the two volumes is the reverse of the chronological order of publication. The part of the theory of the zeta-function here required is what may be called the 'classical' theory, and comprises roughly those properties summarised by Prof. Titchmarsh in his Introduction. This is expounded in detail in the present volume, which is thus complete in itself (apart from a few isolated references to Titchmarsh which do not affect the understanding of the book as a whole); and the relevant parts may serve as an introduction to the more profound study of the zeta-function in the companion volume. The present tract is not intended exclusively for specialists, for whom the more comprehensive treatises of Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* and *Vorlesungen über Zahlentheorie*, are already available; it aims rather at making the subject accessible to a wider circle of readers.

This volume like its companion has its origin in the Bohr-Littlewood manuscript referred to by Prof. Titchmarsh in his preface. This manuscript forms the basis of the present version, but a complete revision was found desirable in order to bring the work up-to-date and to take account of improvements of technique introduced since the preparation of the original. In the task of revision I derived much assistance from lecture notes kindly placed at my disposal by Prof. Littlewood. My indebtedness to the two books of Landau already referred to will be too obvious to readers of those works to need special emphasis here. The proof-sheets have been read by Prof. H. Bohr and Prof. J. E. Littlewood, the authors of the original manuscript, and also by Prof. G. H. Hardy, Dr A. Zygmund, Mr R. M. Gabriel, and Mr C. H. O'D. Alexander, and to these my thanks are due for a number of corrections and improvements. To Prof. N. Wiener I am indebted for some valuable comments on the concluding sections of Chapter II.

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