Cambridge University Press
978-0-521-39783-4 - Twistors in Mathematics and Physics
Edited by T.N. Bailey and R.J. Baston
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Twistor Theory After 25 Years — its Physical Status and Prospects

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Introduction

The primary objective of twistor theory originally was—and still is—to find a deeper route to the workings of Nature; so the theory should provide a mathematical framework with sufficient power and scope to help us towards resolving some of the most obstinate problems of current physical theory. Such problems must ultimately include: (1) removing the infinities of quantum field theory, (2) ascertaining the nature and origin of symmetry and asymmetry in the classification of particles and in physical interactions, (3) deriving, from some fundamental principle, the strengths of coupling constants and the masses of particles, (4) finding a quantum gravity theory capable of satisfactorily addressing the issues raised by space-time singularities and the structure of space-time in the small, (5) constructing a picture that makes sense of the puzzling non-locality and conceptual peculiarities inherent in the process of quantum measurement. Does twistor theory have anything of significance to contribute concerning these matters? Might it at least point us in some appropriate directions?

I shall comment on these issues individually in a moment. But as things stand, it must be said that the successes of twistor theory to date have been almost entirely in applications within *mathematics*, rather than in furthering our understanding of the nature of the physical world. I would think of twistor theory's physical role, so far, as being something perhaps resembling that of the Hamiltonian formalism. That formalism provided a change in the framework for classical Newtonian theory rather than a change in Newtonian theory itself. The Hamiltonian scheme (at least Hamilton's own part in its development) was motivated very much by a physical analogy between the behaviour of particles and of waves; but it was not until the advent of quantum physics that a change in physical theory was put forward—indeed one in which particles and waves became actually the same thing, rather than being merely analogous. When the mathematics for a quantum theory was required, Hamiltonian formalism was in place and provided the ideal vehicle, ready to accommodate the essential changes that were needed, in order that physical theory could be transported from classical to quantum. The ambitious role set out for twistor theory, then, is that, likewise, when enough of its mathematics has been developed, that theory also will be in place and, with relatively minor changes, will turn out to be just what is required for a much needed new physics.

In this article, I shall be concerned primarily with physical issues, and how I feel that twistor theory stands, or ought to stand, with regard to them. The *mathematical* applications of the theory are well covered by articles by other authors in this volume, and some of these applications have proved to be unexpectedly fruitful. With regard to *physical* applications and aside from developments connected with the fundamental issues referred to above, which I discuss in a moment), there has one noteworthy and *unanticipated* success: the concept of *quasi-local mass and (angular) momentum in general relativity*. For a great many years, relativists had resigned themselves to the idea that the mass-energy of the gravitational field cannot be localized, and only the *total* energy of an asymptotically flat space-time can be assigned an unambiguous meaning. Twistor theory now allows us to do a good deal better (Penrose 1982, Penrose and Rindler 1986), though various difficulties remain. A full and up-to-date account is to be found in Paul Tod's article (1990—this volume; see also Mason and Frauendiener (1990), this volume), and it will not be necessary for me to go into the details here.

Nonetheless, one is compelled to confess that, so far, rather little that is both tangible and new has come through with regard to twistor theory's *original* physical aspirations. Let us now try to see how the theory stands with regard to each in turn of the above-mentioned questions.

1 The infinities of quantum field theory

Of the fundamental physical problems referred to in the opening paragraph, it is only the issue of infinities of quantum field theory that has been significantly addressed, so far. The main progress in this direction has been in the theory of *twistor diagrams* (see Hodges 1990 and Huggett 1990, this volume) which has been evolved as the twistor analogue of Feynman graphs. The intention has been that a procedure essentially equivalent to the conventional Feynman theory could be developed—except that it is intended that *finite* answers are to be obtained in important cases where the Feynman graphs diverge. The initial work in this area proceeded to a considerable extent by guesswork, analogy, geometrical considerations, aesthetics, and wishfull thinking (Penrose and MacCallum 1972, Penrose 1975b; cf. also Sparling 1975, Qadir 1978), but then later work (Hodges 1983a,b, 1985a,b 1990, this volume, cf. also Huggett 1990 this volume) not only put the twistor diagram theory on a sound basis, but also led to actual changes in which certain of the infinities of the standard theory have indeed become replaced by finite expressions.

As an initial step of the original scheme, the usual momentum states of the conventional theory are replaced by the (finite-normed) *elementary states* that arise naturally in twistor theory, so that the calculated amplitudes have some chance of being actually finite, rather than involving delta-functions, as is the case with momentum states. This allows, as a general objective of the twistor diagram formulation, that amplitudes might be computed over *compact* (high-dimensional) contours, the integrands being supposed to be analytic expressions at all points of the contours, so that the answers would accordingly be always guaranteed to be finite whenever these two requirements can be satisfied. However, since some of the answers, as 'correctly' computed by Feynman graph methods, are actually divergent, this entails that some change must be introduced into the procedures from those that would be obtained by direct translation of the corresponding Feynman graphs. This applies, as Andrew Hodges noted a good many years ago, even to some 'tree diagrams' of the standard Feynman theory, which are infra-red divergent. He was able to circumvent this problem in an ingenious way (Hodges 1985a), by replacing the previous factors $Z^\alpha W_\alpha$ that had ocurred in twistor diagram expressions according to

$$Z^\alpha W_\alpha \longmapsto Z^\alpha W_\alpha + k$$

where k is some (dimensionless) numerical constant whose value would be ultimately fixed by theory or experiment. At first, k merely provides a number whose logarithm enters into a finite expression which replaces each infra-red divergent quantity (the divergence being recovered when $k \to 0$), but k seems also to play a key role in eliminating ultra-violet divergences (Hodges 1985a) and it has a separate importance in relation to twistor diagrams for *massive* particles.

The factors $Z^\alpha W_\alpha + k$ bear some resemblance to factors

$$Z^\alpha X^\alpha I_{\alpha\beta} + m,$$

which Hodges uses in the twistor diagrams describing massive particles ('projection operators' for the mass eigenvalue m). These diagrams also make use of the so-called 'universal bracket factor' $[...]_U$ (cf. Penrose 1979c, Hodges 1985b) which is 'defined' (formally) by the divergent expression

$$[x]_U = \ldots + (x)_{-2} + (x)_{-1} + (x)_0 + (x)_1 + (x)_2 + \ldots$$

where, for $n = 0, 1, 2, 3, \ldots$

$$(x)_{-n} = (x)^{-n}/n! \quad \text{(contour with boundary on x=0)}$$

and

$$(x)_{n+1} = -n!(-x)^{-n-1}/2\pi i \quad \text{(contour surrounding x=0)},$$

or by the formal expression, suggested by George Sparling (cf. Penrose 1979)

$$\oint [x]_U = \left\{ \oint \int_0^\infty + \not\oint \oint \right\} \frac{e^z}{z+x} dz$$

Although the divergence difficulties are cleverly circumvented in Hodges's particular 'mass-projection' expression, a proper general understanding of the universal bracket is still lacking.

Many other problems in twistor diagram theory remain to be solved, but some good progress has been made towards the goal of finding a complete formulation of the *standard model* of weakly or strongly interacting particles in twistor diagram terms (Hodges 1990, this volume). One particular problem is to obtain a fuller understanding of the (high-dimensional) contours that occur in twistor diagrams. As noted above, these are supposed always to be compact (perhaps with boundary) so that the integrals will always be finite. The extent to which it has been possible to satisfy this compactness requirement so far has been definitely encouraging. However, particularly when mass is present, the status of this requirement is still unclear and it seems to demand the use of 'blown-up twistor space' according to which the line I, in projective twistor space $\mathbf{PT}$, is replaced by a quadric surface. This is related to the 'googly twistor space' needed for the description of general relativity, and which will be described in outline below.

Another particularly important issue of twistor diagram theory is to understand, in purely twistorial terms, which twistor diagrams are to appear in any given process, and with what weighting (and sign) each diagram is to occur. A popular approach to the corresponding problem for Feynman diagrams, in modern quantum field theory, is to use the (formal) method of *path integrals.* However, an analogous procedure for twistor theory has not yet come to light. There is an apparently fundamental conflict between the twistor description of fields and that which is addressed by a path-integral approach. In the latter, paths are deliberately allowed in which the field equations are violated, whereas in twistor theory it is considered to be a virtue for the classical field equations to come out as solved *automatically* by the twistor descriptions! In twistor (diagram) theory 'off-shell' contributions in which field equations are violated come about in a different way (in effect, by the introduction of further twistors). The relation between this and path integrals has not yet come to light.

Another possible line of approach to the problem of 'twistor diagram generation' is through the ideas of a generalized conformal field theory involving 'pretzel twistor spaces' (Hodges, Singer and Penrose 1989, Penrose 1989), though this approach has not yet progressed very far. A 'pretzel twistor space' is a higher-dimensional analogue of a Riemann surface (with 'holes'), as occurs in standard conformal field theory (or string theory). Though perhaps superficially similar to a 'membrane' (or 'p-brane') theory (i.e. higher-dimensional string theory), this approach is fundamentally different in that the generalizations of Riemann surfaces are *complex manifolds*, either of three dimensions (projective case) or of four (non-projective), and in that the complex manifolds play the role of *twistor* spaces, in relation to space-time, rather than being regarded as being 'in' space-time. So long as general relativity is not involved, each such 'pretzel twistor space' would be a *flat twistor space* X, defined by the following properties, in the three-dimensional projective case:

1. X is a compact complex 3-manifold with boundary ∂X;
2. each component of ∂X is a copy of (i.e. is CR-equivalent to) the $\mathbf{PN}$ of standard twistor theory;
3. each point of $X - \partial X$ has a neighbourhood which is holomorphic to a neighbourhood of a line in $\mathbf{CP}^3$;
4. the canonical bundle of X admits a fourth root.

The reason for condition (4) is to enable the corresponding non-projective (four-dimensional) flat twistor space to be defined as the appropriate line bundle over the projective flat twistor space.

According to this proposal, a scattering process would be described by one of these flat twistor spaces (or by a linear superposition of processes described by different such spaces), where a positive or negative orientation would be assigned to each component of ∂X. Each positively oriented component would refer to an incoming particle state, and each negatively oriented component, to an outgoing particle state, where the in- and out-states (taken to be massless, in the first instance) would be described in the standard way by (1st) sheaf cohomology elements (restricted to $\mathbf{PN}$). The procedure follows closely the one adopted in ordinary conformal field theory (cf. Segal 1990, Witten 1989). It is strongly motivated by the close analogy between the way (a) that the 'equator' S^1 (unit circle) divides the Riemann sphere into the 'northern hemisphere' S^+ and 'southern hemisphere' S^- and the way (b) that $\mathbf{PN}$ divides $\mathbf{PT}$ into $\mathbf{PT}^+$ and $\mathbf{PT}^-$. The splitting (a) of functions (i.e. H^0-elements) on S^1 into their positive and negative frequency parts according to whether they extend holomorphically into S^+ or S^-, is closely mirrored by the splitting (b) of solutions of the massless field equations into their positive and negative frequency parts according to whether the corresponding twistor functions (as H^1-elements) extend into $\mathbf{PT}^+$ or $\mathbf{PT}^-$. (This important fact realized one of the key original motivations behind twistor theory; cf. Penrose 1986a.)

The hope is that there should be some close relation between the construction of flat twistor spaces, their corresponding conformal field theories, and twistor diagrams. This would mirror the way that the early string theory showed how duality diagrams (Riemann surfaces with 'holes') could be used to make sense of the 'counting' of Feynman diagrams in strong interaction theory, and can also serve to replace certain collections of infinite Feynman diagrams by finite expressions. There does seem to be a corresponding role for twistor diagrams in relation to flat twistor spaces, but unfortunately this has not been explored very far as yet.

In conformal field theory, the in-states [out-states] are elements of Fermionic 'Fock spaces'

$$\mathcal{H} = \mathbf{C} \oplus H \oplus H \wedge H \oplus H \wedge H \wedge H \oplus H \wedge H \wedge H \wedge H \oplus \ldots$$

where each H is the space of positive-frequency [negative-frequency] functions (or sections of bundles) on a positively [negatively] oriented S^1 that constitutes a 'hole' boundary in the Riemann surface in question. In the case of a pretzel twistor space, the 'hole' boundaries are copies of $\mathbf{PN}$, and instead of functions, we have first cohomology elements, representing wave-functions of massless fields. We cannot interpret the elements of the higher-order spaces

$$H \wedge H, \quad H \wedge H \wedge H, \quad H \wedge H \wedge H \wedge H, \quad \ldots$$

as representing many-particle states, since each different particle taking part in a scattering process is to be represented by a different $\mathbf{PN}$ hole. Instead, the elements of these higher-order spaces (functions of several twistors) must presumably represent massive particles, in accordance with the twistor particle programme, that will be briefly described in the next section.

2 Symmetry and asymmetry in particle interactions

One of the most striking things about the twistor formulation—for good or for bad—is that by choosing the twistor space $\mathbf{PT}$ to be primary, rather than the dual space $\mathbf{PT}^*$ (or *vice versa*), we are led to an essentially *left-right asymmetric* description of physics. This would seem to be a desirable thing when we are trying to describe aspects of physics—notably weak interactions—for which such left-right asymmetry is known to be a fact of nature, but its desirability is more questionable for those interactions which are believed to be left-right *symmetric*. In particular, in the case of general relativity, we have a fundamental theory of space-time structure which is left-right symmetric, and this presents a severe challenge to any asymmetric twistorial description. It is a remarkable fact, however, that several new approaches to the description of standard general relativity have also been guided, for apparently quite independent reasons, into a left-right asymmetric formulation. These are approaches which relate, in one way or another, to what are known as 'Ashtekar variables' (Ashtekar 1988—see also Mason & Frauendiener 1990, this volume). The left-right asymmetry is expressed as an asymmetry between primed and unprimed 2-spinor indices—or, what amounts to the same thing, to an asymmetry between anti-self-dual and self-dual curvatures.

The basic 'twistorial' reason for believing in a left-right asymmetric approach to physics (i.e. a preferance either for $\mathbf{PT}$ or for $\mathbf{PT}^*$ in the formulation) arises from the 'twistor-function' description of linear massless fields. A holomorphic function $f(Z^\alpha)$ (actually a representative 1-cocycle for an element of $H^1(\mathbf{PT}^+, \mathcal{O})$), which is homogeneous of degree $-n-2$, describes a wave function for a massless particle of helicity $n/2$ (n being an arbitrary integer). It is a striking fact that in this way we can automatically incorporate the two essential requirements for a massless one-particle wave-function, namely satisfaction of both the massless field equation and the positive-frequency condition. The fact that we are using a *holomorphic* function of the twistor Z^α (i.e. a 'function of Z^α' rather than a 'function of Z^α and $\overline{Z}_\alpha$') is the twistorial version of the basic quantum-mechanical requirement that ordinary wave functions must be functions just of position (or just of momentum) not functions of both position and momentum. For ordinary wave functions we can, if we prefer, choose functions of momentum instead of functions of position for our descriptions of quantum states, so long as we are consistent about this. Likewise, we can, if we prefer, consistently use holomorphic functions of *dual* twistors W_α (i.e. holomorphic functions of $\overline{Z}_\alpha$, where we simply relabel $\overline{Z}_\alpha$ as W_α, i.e. *anti*-holomorphic functions of Z^α. However, such an alternative choice must be consistent: it would make no sense to use, say, a position description for particles of positive electric charge and a momentum description for particles of negative electric charge; and likewise it would make no sense to use, say, a dual twistor description for massless particles of positive helicity and a twistor description of massless particles of negative helicity. (Such a description might have seemed tempting in view of the fact that right-handed—i.e. self-dual—non-linear gravitons seem to have a natural description in terms of dual twistors and left-handed—i.e. anti-self-dual—non-linear gravitons, a natural description in terms of twistors.) A particularly awkward aspect of any such attempt to describe massless particles in this hybrid way arises from the fact that there is often the need to describe massless particles which are not simply entirely right-handed or left-handed, such as plane-polarized photons. Thus, at least if we are describing massless particles, it seems to be necessary to make a choice in our twistorial representation: *either* a description in terms of twistors Z^α must be used *or* a description in terms of dual twistors W_α.

So long as we are concerned only with *linear* massless fields (without sources), this does not imply any serious left-right asymmetry for what it is possible to achieve with the twistor formalism, but, as is apparent with the the 'non-linear graviton construction' for (anti-)self-dual gravitational fields (Penrose 1976) and the Ward construction for (anti-)self-dual Yang-Mills fields (Ward 1977), the situation seems very awkwardly different for non-linear fields. (This raises the issue of the 'googly problem' which I shall return to later.) If it is supposed that Nature's ways actually accord with some of the basic ideas of twistor theory, and that she thus prefers, say, a twistorial description—or else she prefers a dual-twistorial description—then it would be expected that some left-right asymmetry should be present with the actual physics of non-linear massless fields. Of course, we already know that weak interactions are left-right asymmetric, but the above considerations should apply also to the gravitational field. The standard Weinberg-Salam-Glashow-Ward theory of unified electromagnetic and weak interactions implies that there is an indirect left-right asymmetry in electromagnetism, but the above considerations seem to imply a 'twistor expectation' of a left-right asymmetry in gravitation also.

Even for linear fields, there is twistorial left-right (or, rather, a self-dual/anti-self-dual) asymmetry in the case of fields with *sources*. For example, in the case of a Coulomb field, the twistor-function for the self-dual part would have the form

$$f(Z^\alpha) = \frac{1}{(Q_{\alpha\beta}Z^\alpha Z^\beta)^2}$$

while that for the anti-self-dual part would be something like

$$g(Z^\alpha) = \log(Q_{\alpha\beta}Z^\alpha Z^\beta)$$

or

$$g'(Z^\alpha) = \log\frac{Q_{\alpha\beta}Z^\alpha Z^\beta}{A_\gamma Z^\gamma B_\delta Z^\delta}.$$

In none of these cases do we get a global representation of the space-time field as an H^1 element in twistor space, but in the self-dual case we get a global description as a relative H^1 element (Bailey 1985). In the anti-self-dual case this does not seem to be so, however, and the situation is more obscure. Moreover, when we go over to the 'non-linear' Ward representation in terms of a line-bundle over in the anti-self-dual case we get 'charge quantization' and a non-Hausdorff bundle (Penrose and Sparling 1979, Bailey 1985). There is no analogue (as yet) in the self-dual case.

I have phrased the above discussion in terms of left-right asymmetry (i.e. parity P), since this is the most obvious of the discrete symmetry operations which convert left-handed massless particle into right-handed ones. However the operation C of charge-conjugation (particle-antiparticle interchange) also achieves this (witness the case of a neutrino), as do the operations CT and PT (where T stands for time-reversal symmetry). All of these four symmetries are violated in weak interactions, and it would appear that such violations could be well accommodated by twistor theory. In the context of the rules governing twistor diagrams, one only needs an asymmetry under interchange of black spots (twistors Z^α) with white spots (dual twistors W_α). But in addition, T and CP are known to be violated in K_0-decay, and this could arise, twistorially, out of some sort of asymmetry between $\mathbf{PT}^+$ and $\mathbf{PT}^-$. There are (controversial!) reasons for believing that even CPT should be violated in quantum gravity theory (cf. Penrose 1981, 1986b, 1989 and below). Such further symmetry violations would seem much more natural in the context of twistor theory than they do in standard space-time descriptions, but a good deal more understanding is needed if we are to see what the exact role of twistor theory in symmetry violation actually is.

Let us next consider massive particles, and how the quantum description of such particles is best to be incorporated into twistor theory. We recall that the twistor theory of massless fields is closely bound up with the quantized expressions for momentum and angular momentum for a massless particle:

$$p_a = \pi_{A'}\overline{\pi}_A, \quad M^{ab} = i\omega^{(A}\overline{\pi}^{B)}\epsilon^{A'B'} - i\overline{\omega}^{(A'}\pi^{B')}\epsilon^{AB}$$

where, in the twistor (as opposed to dual twistor) representation of massless wave-functions, we make the replacements

$$\overline{\pi}_A \longmapsto -\frac{\partial}{\partial\omega^A}, \quad \overline{\omega}^{A'} \longmapsto -\frac{\partial}{\partial\pi_{A'}},$$

in accordance with the standard twisor quantization rule

$$\overline{Z}_\alpha \longmapsto -\frac{\partial}{\partial Z^\alpha}.$$

(Here I am taking $\hbar = 1$.) These give the standard momentum and angular momentum operators when applied to a twistor function of the one twistor variable Z^α, the squared mass $m^2 = p_a p^a$ being identically zero. The procedure which is adopted in twistor particle theory in order to handle particles of *non*-zero mass is to replace the above expressions by sums

$$p_a = \sum_{i=1}^{r} \pi_{iA'}\overline{\pi}_{iA} \quad M^{ab} = \sum_{i=1}^{r}\{i\omega_i^{(A}\overline{\pi}_i^{B)}\epsilon^{A'B'} - i\overline{\omega}_i^{(A'}\pi_i^{B')}\epsilon^{AB}\}$$

where instead of acting on functions of just one twistor, these (quantized) operators now act on functions of several twistors

$$Z_1^\alpha, \ldots, Z_r^\alpha,$$

with

$$Z_i^\alpha = (\omega_i^A, \pi_{iA'}).$$

A twistor (wave-)function for a massive particle is now to be a holomorphic function of $Z_1^\alpha, \ldots, Z_r^\alpha$ (although the possibility that certain of these twistor variables might better be taken as dual twistors should not be overlooked). According to the original twistor-particle scheme (which I sometimes refer to as 'naïve twistor particle theory') *leptons* were to be the particles described by functions of just *two* twistors, say Y^α and Z^α, and *hadrons* by functions

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of *three* twistors, say X^α, Y^α and Z^α. One of the reasons for this was that the n-twistor internal symmetry group (the group of linear transformations of Z_i^α and $\overline{Z}_{i\alpha}$ that leaves p_a and M_{ab} invariant) is a slight inhomogeneous extension of $\mathrm{SU}(2)\times\mathrm{U}(1)/\mathbb{Z}_2$ (or U(2)) in the case of $n = 2$, and a rather larger inhomogeneous extension of SU(3) in the case $n = 3$. These were basically the symmetry groups that arose in the standard classification of leptons and hadrons, respectively—in the 'good old days' before 'charm' was discovered! The 3-twistor scheme for the hadrons of those days provided quite a strikingly good fit, for the most part, although there were some notable anomalous multiplets, such as the nonet (rather than the expected octet) which involves the η^0, and certain families of resonances which seemed to have the wrong symmetries (see Hughston 1979). The 2-twistor scheme for leptons seemed to present more serious problems, since it provided for only two quantum numbers (in addition to spin and mass)—identifiable with charge and lepton number—leaving no way of distinguishing the muon from the electron. An ingenious suggestion due to George Sparling was that the required additional quantum number might be, in effect, the sign of the quantum number for total spin! He noted that the squared total quantum-mechanical spin J_2 of a massive particle, since it takes the form

$$J_2 = j(j+1)$$

(in units of $\hbar$), where j is the usual total spin quantum number, is invariant under

$$j \longmapsto -j-1.$$

Thus, for an observed total spin value, there are really two possible values of the quantum number j, namely j and $-j-1$. Sparling's suggestion was that we can allow for *negative* values of j , and the idea was that perhaps what distinguishes the muon from the electron was that one of these particles has $j = \frac{1}{2}$ and the other has $j = -\frac{3}{2}$. (There was some hint of support for this kind of idea in the expression for the spin operator in the 2-twistor scheme.) It may be that this suggestion has less plausibility now than it had at the time, owing to the further complication of the discovery of the τ-lepton. Nevertheless there does seem to be something in this idea, which shows up in the case of *hadrons*. It is (or was!) well known that if we plot the hadron resonances, for each particular set of values of its SU(3) quantum numbers, in a diagram with the j vertical and m^2 horizontal, then most of the resonances will lie on a family of remarkably straight lines rising off to the right—the *Regge trajectories*. In the case of the baryons, it is possible to divide these trajectories into two classes, those corresponding to natural parity ($\epsilon = 1$) and those corresponding to unnatural parity ($\epsilon = -1$). If we assume that (say) the natural parity baryons have positive j and the unnatural parity ones, negative j, and we now plot m^2 against j, we find that the pairs of Regge trajectories join together into single straight lines,