Chapter 0
PRELIMINARIES

For this manuscript, all groups will be assumed finite. If G is a group and $\mathcal{F}$ an (arbitrary) field, an $\mathcal{F}[G]$-module V will mean that V is a right $\mathcal{F}[G]$-module and that V is finite dimensional over $\mathcal{F}$. Recall that V is *completely reducible* if V is the sum of simple $\mathcal{F}[G]$-modules. In this case, V is actually a direct sum of simple modules. Indeed, if $V \neq 0$ is completely reducible, then $V = V_1 \oplus \cdots \oplus V_l$ where $V_i \neq 0$ is the direct sum of simple isomorphic $\mathcal{F}[G]$-modules and if W_i and W_j are simple submodules of V_i and V_j (resp.), then $W_i \cong W_j$ (as $\mathcal{F}[G]$-modules) if and only if $i = j$. Then V_i are called the *homogeneous components* of V and are unique (not merely up to isomorphism, but the V_i are unique submodules). Now $V_1 = U_1 \oplus \cdots \oplus U_t$ for isomorphic $\mathcal{F}[G]$-modules U_i. While t is unique, the U_i are unique only up to isomorphism.

Because solvable groups have an abundance of normal subgroups, we begin by recalling Clifford's Theorem:

0.1 Theorem. *Suppose that V is an irreducible $\mathcal{F}[G]$-module and $N \trianglelefteq G$. Then*

(a) *V_N is completely reducible and so $V_N = V_1 \oplus \cdots \oplus V_l$ where the V_i are the homogeneous components of V_N;*

(b) *G/N transitively permutes the V_i by right multiplication;*

(c) *If W_i and W_j are irreducible N-submodules of V_i and V_j (resp.), then $\dim(W_i) = \dim(W_j)$ for all i, j; and*

(d) *If $I = \{g \in G \mid V_1 g = V_1\}$ is the inertia group in G of V_1, then V_1 is an irreducible I-module and $V \cong V_1^G$ (induced from I to G).*

Proof. This is Hauptsatz V, 17.3 of [Hu]. □

0.2 Proposition. *Suppose that V is an irreducible $\mathcal{F}[G]$-module, $N \trianglelefteq G$ and V_N is not homogeneous.*

(i) *If $C \trianglelefteq G$ is maximal such that V_C is not homogeneous, then G/C faithfully and primitively permutes the homogeneous components of V_C.*

(ii) *There exists $N \leq D \triangleleft G$ such that $V_D = W_1 \oplus \cdots \oplus W_s$ for D-invariant W_i that are faithfully and primitively permuted by G/D $(s > 1)$. Furthermore, whenever $N \leq L \leq D$ with $L \trianglelefteq G$, V_L is not homogeneous and each W_i is a sum of homogeneous components of V_L.*

Proof. Write $V_N = V_1 \oplus \cdots \oplus V_t$ where the V_i are the homogeneous components of V_N. Suppose that $N \leq M \trianglelefteq G$ and $W = V_1 \oplus \cdots \oplus V_s$ is M-invariant. We claim that W is a direct sum of homogeneous components of V_M. To see this, let X and Y be isomorphic irreducible M-submodules of V with $X \leq W$. Now X_N and Y_N have isomorphic irreducible submodules X_0 and Y_0 (resp.). Since the V_i are homogeneous components of V_N, X_0 and Y_0 are contained in the same V_i. Thus $Y_0 \leq W$ and $Y \cap W \neq 0$. Then $Y \leq W$, establishing the claim.

(i) Now G transitively permutes the homogeneous components of V_C. Let K be the kernel of this permutation action, so that $C \leq K \trianglelefteq G$. Applying the last paragraph to V_C, each homogeneous component of V_C is a direct sum of homogeneous components of V_K. By maximality of C, $C = K$, proving that G/C acts faithfully on the homogeneous components of V_C. This action is primitive by the first paragraph and choice of C.

(ii) Since G transitively permutes $\Omega = \{V_1, \ldots, V_t\}$, we may write $\Omega = \Delta_1 \,\dot\cup \cdots \dot\cup\, \Delta_s$ with $s > 1$ and G primitively permuting $\{\Delta_1, \ldots, \Delta_s\}$. In other words, $V_N = W_1 \oplus \cdots \oplus W_s$ where $s > 1$ and each W_i is a sum of some homogeneous components of V_N and such that G primitively permutes the

W_i. Let D be the kernel of the permutation action of G on $\{W_1, \dots, W_s\}$. Then $V_D = W_1 \oplus \cdots \oplus W_s$ for D-invariant W_i that are faithfully and primitively permuted by G/D. Furthermore, whenever $N \leq L \leq D$ with $L \trianglelefteq G$, each $(W_i)_L$ is a sum of homogeneous components of V_L, by the first paragraph. Since $s > 1$, V_L is not homogeneous. □

The structure of solvable primitive permutation groups is well-known and discussed below in Section 2. In particular, a nilpotent and primitive permutation group has prime order (see [Hu, Satz II, 3.2]).

0.3 Corollary. *Suppose that V is an irreducible G-module, $N \trianglelefteq G$ and V_N is not homogeneous. If G/N is nilpotent, there exists $N \leq C \triangleleft G$ with $|G : C| = p$, a prime such that $V_C = V_1 \oplus \cdots \oplus V_p$ for homogeneous components V_i of V_C.*

0.4 Proposition. *Suppose that V is an irreducible $\mathcal{F}[G]$-module and that $\mathcal{K}$ is an extension field of $\mathcal{F}$.*

(i) *If $\operatorname{char}(\mathcal{F}) \neq 0$, then $V \otimes_{\mathcal{F}} \mathcal{K} = W_1 \oplus \cdots \oplus W_l$ for non-isomorphic irreducible $\mathcal{K}[G]$-modules W_i.*

(ii) *If $\mathcal{K}$ is a Galois extension of $\mathcal{F}$, then $V \otimes_{\mathcal{F}} \mathcal{K} \cong e(V_1 \oplus \cdots \oplus V_t)$ for a positive integer e and non-isomorphic irreducible $\mathcal{K}[G]$-modules V_i. Furthermore the V_i are afforded by representations X_i that are conjugate under $\operatorname{Gal}(\mathcal{K} : \mathcal{F})$. Indeed $\{X_1, \dots, X_t\}$ is a single orbit under $\operatorname{Gal}(\mathcal{K} : \mathcal{F})$.*

Proof. See [HB, Theorems VII, 1.15 and VII, 1.18 (b)]. The $\mathcal{K}[G]$-module $V \otimes_{\mathcal{F}} \mathcal{K}$ is denoted by $V_{\mathcal{K}}$ in [HB] and by $V^{\mathcal{K}}$ in [Is]. □

Suppose V is a faithful irreducible $\mathcal{F}[G]$-module for some field $\mathcal{F}$. If $\mathcal{K}$ is an extension field of $\mathcal{F}$, then G has a faithful irreducible $\mathcal{K}[G]$-module W by Proposition 0.4. By choosing $\mathcal{K}$ to be algebraically closed, G has a faithful absolutely irreducible representation $X : G \to M_n(\mathcal{K})$ for some n. Then the centralizer in $M_n(\mathcal{K})$ of $X(G)$ consists of scalar matrices. If G is abelian, then G must be cyclic and $n = 1$. We thus have the following well-known result which is of particular importance to the structure of quasi-primitive linear groups.

0.5 Lemma. *If an abelian group A has a faithful irreducible module W (over an arbitrary field $\mathcal{F}$), then A is cyclic. If furthermore W is absolutely irreducible, then $\dim_{\mathcal{F}}(W) = 1$.*

The following lemma is sometimes referred to as Fitting's lemma, although [Hu] credits Zassenhaus.

0.6 Lemma. *Suppose G acts on an abelian group A by automorphisms and $(|G|, |A|) = 1$. Then $A = [G, A] \times \mathbf{C}_A(G)$.*

Proof. See [Hu, Satz III, 13.4]. □

We use $\mathrm{Irr}\,(G)$ to denote the set of the ordinary (i.e. complex) irreducible characters of the group G and let $\mathrm{char}\,(G)$ denote the set of all ordinary characters of G. Of course, $\mathrm{char}\,(G) \subseteq \mathrm{cf}\,(G)$, the set of class functions of G, and we let $[\chi, \theta]$ denote the inner product of χ, $\theta \in \mathrm{cf}\,(G)$. For $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}\,(N)$, we let $\mathrm{Irr}\,(G|\theta) = \{\chi \in \mathrm{Irr}\,(G) \mid [\chi_N, \theta] \neq 0\}$. By Frobenius reciprocity, $\mathrm{Irr}\,(G|\theta)$ is the set of irreducible constituents of the induced character θ^G.

Let $\mathcal{F}$ be a field of characteristic p such that $\mathcal{F}$ contains a $|G|$-th root of unity. Then $\mathcal{F}$ is a splitting field for all subgroups of G (i.e. every irreducible $\mathcal{F}$-representation of every subgroup of G is absolutely irreducible). It is customary to choose $\mathcal{F}$ so that $\mathcal{F}$ is a quotient ring of an integral domain of characteristic zero. This is often done via p-modular systems, as in Section 3.6 of [NT]. A slightly different approach is given in Chapter 15 of [Is]. We should point out here that Chapter 15 of [Is] is only intended as an introduction to modular theory and as such is not complete. Recall that each $g \in G$

has a unique factorization $g = g_p g_{p'} = g_{p'} g_p$ where g_p is a p-element and $g_{p'}$ is p-regular (i.e. $p \nmid o(g_{p'})$). Each irreducible $\mathcal{F}$-character χ of G can then be lifted to a complex-valued function φ, defined on p-regular elements of G. Now φ is called an *irreducible Brauer character* of G, the set of which is denoted $\mathrm{IBr}_p(G)$. (Actually there are some choices involved in this procedure, but it is usual to do this simultaneously for all irreducible representations of all subgroups of G to avoid complications). Because $\chi(g) = \chi(g_{p'})$ for all $g \in G$, defining the lift $\varphi \in \mathrm{IBr}_p(G)$ only on p-regular elements loses no information and avoids technical difficulties. Now there is a 1–1 correspondence between $\mathrm{IBr}_p(G)$ and the irreducible $\mathcal{F}$-representations. Indeed, if $\varphi \in \mathrm{IBr}_p(G)$ corresponds to the $\mathcal{F}$-representation afforded by an $\mathcal{F}[G]$-module V, then $\varphi(1) = \dim(V)$. Also $\mathrm{IBr}_p(G)$ is linearly independent over $\mathbb{C}$ and $|\mathrm{IBr}_p(G)|$ is the number of p-regular classes of G.

Let $N \trianglelefteq G$ and $\varphi \in \mathrm{IBr}_p(N)$. We write $\mathrm{IBr}_p(G|\varphi) = \{\beta \in \mathrm{IBr}_p(G) |\ \varphi$ is a constituent of $\beta_N\}$. Now the induced character φ^G is a positive $\mathbb{Z}$-linear sum $\sum a_i \mu_i$ of irreducible Brauer characters μ_i, even though the corresponding induced module may not be completely reducible. By Nakayama reciprocity [HB, Theorem VII, 4.13 (a)] and Clifford's Theorem 0.1, each $\chi \in \mathrm{IBr}_p(G|\varphi)$ is a constituent of φ^G. When G/N is a p'-group, we get the converse and more.

0.7 Proposition. *Suppose that G/N is a p'-group, that $\varphi \in \mathrm{IBr}_p(N)$ and $\theta \in \mathrm{IBr}_p(G)$. Then the multiplicity of φ in θ_N equals the multiplicity of θ in φ^G.*

Proof. Let $\mathcal{F}$ be a splitting field for N and G in characteristic p. Let V be an (irreducible) $\mathcal{F}(G)$-module affording θ, and W an (irreducible) $\mathcal{F}(N)$-module affording φ. Now V_N is completely reducible by Clifford's Theorem. Since G/N is a p'-group and W an irreducible N-module, indeed W^G is completely reducible (see [HB, VII, 9.4]). With both V_N and W^G completely reducible and $\mathcal{F}$ a splitting field for N and G, it follows from Nakayama reciprocity ([HB, VII, 4.13]) that the multiplicity of W as a composition factor of V_N equals the multiplicity of V as a composition factor of W^G.

The proposition now follows. □

0.8 Theorem. *Let* $N \trianglelefteq G$ *and* $\varphi \in \mathrm{IBr}_p(N)$. *If* $I = I_G(\varphi)$, *then* $\psi \to \psi^G$ *is a bijection from* $\mathrm{IBr}_p(I|\varphi)$ *onto* $\mathrm{IBr}_p(G|\varphi)$.

Proof. For ordinary characters, this is Theorem 6.11 of [Is]. More generally, a similar proof works here. Let $\chi \in \mathrm{IBr}_p(G|\varphi)$. Clifford's Theorem 0.1 (d, a) shows that $\chi = \mu^G$ for some $\mu \in \mathrm{IBr}_p(I|\varphi)$ and that $\chi_I = \mu + \Lambda$ for a (possibly zero) Brauer character Λ of I with no irreducible constituent of Λ lying in $\mathrm{IBr}_p(I|\varphi)$.

Let $\psi \in \mathrm{IBr}_p(I|\varphi)$. By Nakayama reciprocity [HB, VII, 4.13 (a)], there exists $\gamma \in \mathrm{IBr}_p(G)$ such that ψ is a constituent of γ_I. But then $\gamma \in \mathrm{IBr}_p(G|\varphi)$ and the last paragraph implies that ψ is the unique irreducible constituent of γ_I lying in $\mathrm{IBr}_p(I|\varphi)$ and $\psi^G = \gamma$. So $\psi \to \psi^G$ is a 1–1 and onto map from $\mathrm{IBr}_p(I|\varphi)$ onto $\mathrm{IBr}_p(G|\varphi)$. □

Theorem 0.8 applies to ordinary characters too; just choose p so that $p \nmid |G|$.

0.9 Lemma. *Suppose that* $N \trianglelefteq G$, $\varphi \in \mathrm{IBr}_p(G)$ *and* φ_N *is irreducible. Then* $\alpha \to \alpha\varphi$ *is a one-to-one map from* $\mathrm{IBr}_p(G/N)$ *onto* $\mathrm{IBr}_p(G|\varphi_N)$.

Proof. By [HB, Theorem VII, 9.12 (b,c)], note $\alpha\varphi \in \mathrm{IBr}_p(G)$ for each $\alpha \in \mathrm{IBr}_p(G/N)$ and the mapping $\alpha \to \alpha\varphi$ is one-to-one. Let $\mu \in \mathrm{IBr}_p(G|\varphi_N)$. It suffices to show $\mu = \beta\varphi$ for some $\beta \in \mathrm{IBr}_p(G/N)$. We mimic the proof of [HB, Corollary VII, 9.13].

Let $\mathcal{F}$ be an algebraically closed field of characteristic p and V an irreducible $\mathcal{F}[G]$-module affording φ. Since $\mu \in \mathrm{IBr}_p(G|\varphi_N)$, Nakayama reciprocity implies that μ is a constituent of ${\varphi_N}^G$ (see comments preceding Proposition 0.7). Thus μ is afforded by a composition factor of ${V_N}^G \cong V \otimes_{\mathcal{F}} \mathcal{F}(G/N)$ (see [HB, VII, 4.15(b)]). If $0 = U_0 < U_1 \cdots < U_m =$

$\mathcal{F}(G/N)$ is a composition series of the $\mathcal{F}(G)$-module $\mathcal{F}(G/N)$, then for each i $V \otimes_{\mathcal{F}} U_i / V \otimes_{\mathcal{F}} U_{i-1} \cong V \otimes U_i / U_{i-1}$ is irreducible, again by [HB, VII, 9.12(b)]. Thus, by the Jordan–Hölder theorem, μ is afforded by $V \otimes_{\mathcal{F}} W$ for an irreducible $\mathcal{F}(G/N)$-module W. So $\mu = \beta\varphi$ for some $\beta \in \mathrm{IBr}_p(G/N)$. □

In presenting Gallagher's Theorem (Lemma 0.9 for ordinary characters), Issacs [Is, Theorem 6.16] first proves the following stronger result under the assumption that φ is G-invariant.

0.10 Lemma. *Suppose that $N \trianglelefteq G$, that $\varphi, \theta \in \mathrm{Irr}(N)$ and $\theta = \chi_N$ for some $\chi \in \mathrm{Irr}(G)$. Assume also that $\varphi\theta \in \mathrm{Irr}(N)$ and $I_G(\varphi) = I_G(\varphi\theta)$. Then $\sigma \to \sigma\chi$ is a bijection from $\mathrm{Irr}(G|\varphi)$ onto $\mathrm{Irr}(G|\varphi\theta)$.*

Proof. Let $I = I_G(\varphi) = I_G(\varphi\theta)$. For $\delta \in \mathrm{char}(I)$, observe that $(\delta\chi_I)^G = \delta^G\chi$ (see [Hu, V, 16.8] or [Is, Ex 5.3]). Now Theorem 6.16 of [Is] yields that $\alpha \to \alpha\chi_I$ is a bijection from $\mathrm{Irr}(I|\varphi)$ onto $\mathrm{Irr}(I|\varphi\theta)$. Employing the Clifford correspondence (Theorem 0.8), $\alpha \to (\alpha\chi_I)^G = \alpha^G\chi$ is a bijection from $\mathrm{Irr}(I|\varphi)$ onto $\mathrm{Irr}(G|\varphi\theta)$. Since $\alpha \to \alpha^G$ is a bijection from $\mathrm{Irr}(I|\varphi)$ onto $\mathrm{Irr}(G|\varphi)$, the lemma follows. □

To employ the above, we would like conditions sufficient to extend characters. Theorem 0.13 is quite useful, in part due to uniqueness (e.g. see Lemma 0.18).

0.11 Proposition. *Suppose that G/N is cyclic and $\varphi \in \mathrm{IBr}_p(N)$ is G-invariant. Then $\varphi = \beta_N$ for some $\beta \in \mathrm{IBr}_p(G)$.*

Proof. See [HB, Theorem VII, 9.9]. □

0.12 Proposition. *Suppose $N \trianglelefteq G$, $\theta \in \mathrm{Irr}(N)$ and θ extends to P whenever P/N is a Sylow subgroup of G/N. Then θ extends to G.*

Proof. See [Is, Corollary 11.31]. □

Let $\mu \in \operatorname{char}(G)$ and let $X\colon G \to GL(n, \mathbb{C})$ be a representation of G affording μ. For $g \in G$, let $\det(\mu)(g) = \det(X(g))$. This is independent of the choice of X and $\det(\mu)$ is a linear character of G. We let $o(\mu)$ be the order of the linear character $\det(\mu)$ of G. The following theorem can often be combined with Proposition 0.12 to extend characters. Note that $\det(\varphi + \mu) = \det(\varphi)\det(\mu)$, and $\det(\varphi\mu) = (\det\varphi)^{\mu(1)}\det(\mu)^{\varphi(1)}$.

0.13 Theorem. *Suppose that $N \trianglelefteq G$, $\theta \in \operatorname{Irr}(N)$ is G-invariant and $(o(\theta)\theta(1), |G/N|) = 1$. There exists a unique extension $\chi \in \operatorname{Irr}(G)$ of θ satisfying $(o(\chi), |G:N|) = 1$. Also, $o(\chi) = o(\theta)$.*

Proof. See [Is, Corollary 8.16]. □

In Theorem 0.13, we call χ the *canonical* extension of θ to G. The uniqueness in Theorem 0.13 is quite useful, often for inductive purposes. For example, we use it to prove the Fong–Swan Theorems. It is also used in the proof of Lemma 0.18, which guarantees the existence of characters of p'-degree and is helpful in Fong reduction. First however, we look at " Glauberman's Lemma", Glauberman correspondence, and some consequences thereof.

0.14 Lemma (Glauberman). *Suppose that A acts on G by automorphisms and $(|A|, |G|) = 1$. Assume that both A and G act on a set Ω and that G acts transitively on Ω. In addition, suppose that $(\omega g)a = (\omega a)g^a$ for all $a \in A$, $g \in G$, and $\omega \in \Omega$. Then*

(a) *A has fixed points in Ω; and*

(b) *$\mathbf{C}_G(A)$ acts transitively on the set of fixed points of A in Ω.*

Proof. See [Is, Lemmas 13.8 and 13.9]. Note that the hypothesis $(\omega g)a = (\omega a)g^a$ is equivalent to the condition that the semi-direct product GA acts on Ω (consistently with the actions of G and A). □

If a group A acts on G via automorphisms, we let $\operatorname{Irr}_A(G) = \{\chi \in \operatorname{Irr}(G) \mid \chi^a = \chi \text{ for all } a \in A\}$.

0.15 Theorem. *Whenever A acts on G by automorphisms with $(|A|,|G|)=1$ and A solvable, there is a uniquely defined bijection $\rho(G,A) : \mathrm{Irr}_A(G) \to \mathrm{Irr}\,(C)$ where $C = \mathbf{C}_G(A)$ such that*

(i) *If A is a p-group and $\chi \in \mathrm{Irr}_A(G)$, then $\chi\rho(G,A)$ is the unique $\beta \in \mathrm{Irr}\,(C)$ satisfying $[\chi_C, \beta] \not\equiv 0$ (mod p).*

(ii) *If $T \trianglelefteq A$, then $\rho(G,A) = \rho(G,T)\ \rho(\mathbf{C}_G(T), A/T)$.*

Proof. See [Is, Theorem 13.1]. □

By "uniquely defined" above, we mean there is only one such map (indeed, else (ii) would be meaningless) and this map is independent of choices made in the algorithm implied by the theorem. This map is known as the *Glauberman correspondence*. If A acts on G with $(|A|,|G|) = 1$, but A not solvable, then $|G|$ is odd. Isaacs [Is 2] has exhibited a "uniquely defined" correspondence whenever A acts on G, $(|A|,|G|) = 1$, and $|G|$ is odd. Moreover, this agrees with the Glauberman correspondence when both are defined [Wo 2]. The combined map is thus referred as the *Glauberman–Isaacs correspondence.* The following appears in [Is, Theorem 13.29] and has a couple of uses in this section alone.

0.16 Lemma. *Suppose A acts on G with A solvable and $(|A|,|G|) = 1$. Suppose $N \trianglelefteq G$ is A-invariant, $\chi \in \mathrm{Irr}_A(G)$ and $\theta \in \mathrm{Irr}_A(N)$. Then $[\chi_N, \theta] \neq 0$ if and only if $[\chi\rho(G,A)_{N\cap C}, \theta\rho(N,A)] \neq 0$.*

Much of the following lemma is a consequence of Glauberman's lemma above. In fact, no more is required for G solvable or for parts (a), (b) and (c) in the general case. For G non-solvable, parts (d), (e) and (f) also employ the Glauberman correspondence and more. All parts appear somewhere (possibly as exercises) in Chapter 13 of [Is]. Due to its importance here (particularly when G is solvable), we give a sketch.

0.17 Lemma. *Suppose that $N \leq G \trianglelefteq \Gamma$ with $(|\Gamma : G|, |G : N|) = 1$ and $N \trianglelefteq \Gamma$. Let $H/N \leq \Gamma/N$ with $G \cap H = N$ and $\Gamma = GH$. Suppose that*

$\theta \in \mathrm{Irr}(G)$ is Γ-invariant and $\varphi \in \mathrm{Irr}(N)$ is H-invariant. Then

(a) *θ_N has an H-invariant irreducible constituent α;*

(b) *If $\mathbf{C}_{G/N}(H/N) = 1$, then α is unique;*

(c) *If $\mathbf{C}_{G/N}(H/N) = G/N$, then every irreducible constituent of θ_N is H-invariant;*

(d) *φ^G has a Γ-invariant irreducible constituent η;*

(e) *If $\mathbf{C}_{G/N}(H/N) = 1$, then η is unique; and*

(f) *If $\mathbf{C}_{G/N}(H/N) = G/N$, then every irreducible constituent of φ^G is Γ-invariant.*

Proof. (a, b, c). By Clifford's Theorem, G/N transitively permutes the set X of irreducible constituents of θ_N. Also H/N permutes X and acts on G/N. Since $(|H/N|, |G/N|) = 1$, Glauberman's Lemma 0.14 applies: some element of X is H/N invariant and $\mathbf{C}_{G/N}(H/N)$ transitively permutes the H-invariant elements of X. Parts (a), (b) and (c) follow.

(d, e, f). Arguing by induction on $|G/N| = |\Gamma : H|$ we may assume without loss of generality that G/N is a chief factor of Γ (note, for part (e), part (a) is employed along with the inductive hypothesis). Let $I = I_\Gamma(\varphi)$, so that $H \leq I \leq \Gamma$ and $I = (I \cap G)H$. Now $I \cap G = I_G(\varphi)$ is H-invariant. Also $\xi \to \xi^G$ gives a bijection from $\mathrm{Irr}(I \cap G|\varphi)$ onto $\mathrm{Irr}(G|\varphi)$ and furthermore ξ is H-invariant if and only if ξ^G is H-invariant. The result follows by induction should $I \cap G < G$. Thus $G \leq I$ and φ is Γ-invariant.

First suppose that G/N is abelian. The group $\mathrm{Irr}(G/N)$ of linear characters acts on $\mathrm{Irr}(G|\varphi)$ by multiplication and this action is transitive (see [Is, Exercise 6.2]). Now $\Gamma/G \cong H/N$ acts on both $\mathrm{Irr}(G/N)$ and $\mathrm{Irr}(G|\varphi)$. Here Glauberman's Lemma 0.14 yields (d), (e) and (f).

Finally we may assume that G/N is non-abelian and thus non-solvable. Since $I_\Gamma(\varphi) = \Gamma$, we may in fact assume that $N \leq \mathbf{Z}(\Gamma)$ (see [Is, Theorem 11.28]). Then there exists $Z \leq N$ and $G_1, N_1 \trianglelefteq \Gamma$ such that $G = G_1 \times Z$, $N = N_1 \times Z$ and $(|H/N|, |G_1|) = 1$. We may write $\varphi = \varphi_1 \times \lambda$ (uniquely)