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History and background

1.1 General introduction

In recent years there has been a remarkable renaissance in the interaction between geometry and physics. After a long fallow period in which mathematicians and physicists pursued apparently independent paths their interests have now converged in a striking manner. However, it appears that parallel problems were being investigated in the past but a common language and framework were missing. This has now been rectified with gauge theory (alias the theory of connections) providing the common ground.

In earlier periods geometry and physics interacted at the classical level, as in Einstein's theory of general relativity, with gravitational force being interpreted in terms of curvature. The new feature of the present interaction is that quantum theory is now involved and it turns out to have significant relations with topology. Thus geometry is involved in a global and not purely local way.

A somewhat surprising feature of the new developments is that quantum field theory seems to tie up with deep properties of low-dimensional geometry, i.e. in dimensions 2, 3 and 4 [3]. Thus the exciting new results of Donaldson [10] on four-dimensional manifolds, and the associated theory of Floer [13] on three-dimensional manifolds, are intimately linked to Yang–Mills theory. This has been made even clearer by Witten [35], where the Donaldson–Floer theory is interpreted as a *topological* quantum field theory in $3+1$ dimensions.

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A slightly different case arises from the recently discovered polynomial invariants of knots by Vaughan Jones [17]. These are related to physics in various ways but the most fundamental is due to Witten [36] who has shown that the Jones invariants have a natural interpretation in terms of a topological quantum field theory in 2+1 dimensions. My purpose in these lectures is to present this new theory of Witten. Shortage of time and the present novelty and incompleteness of the theory mean that this is not a definitive treatment. Rather it is an introduction to Witten's ideas, presented from the mathematical point of view. The whole subject is still developing rapidly and a provisional account accessible to mathematicians may serve a useful purpose.

1.2 Gauge theories

The prototype of all gauge theories is electromagnetism. From the geometrical point of view the electromagnetic potential a_μ ($\mu = 1, \ldots, 4$) defines a connection for a $U(1)$ bundle over Minkowski space M. The field is the corresponding curvature

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \quad (\partial_\mu = \partial/\partial x_\mu).$$

Maxwell's equations in vacuo take the form

$$\mathrm{d}f = 0, \quad \mathrm{d}^*f = 0$$

where f is now viewed as a 2-form, d is the exterior derivative and d^* is its formal adjoint (relative to the Minkowski metric).

Non-abelian gauge theories are obtained by replacing $U(1)$ with a compact non-abelian Lie group G, e.g. $SU(n)$. A potential is then a connection A over Minkowski space, with components A_μ in the Lie algebra of G, and the field is the curvature F with components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The most straightforward generalization of Maxwell's

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equations are the Yang–Mills equations†

$$dF = 0, \quad d^*F = 0,$$

where d and d* are *covariant* derivatives. Gauge theories possess an infinite-dimensional symmetry group given by functions $g: M \to G$ and all physical, or geometric, properties are gauge invariant.

To specify a physical theory the usual procedure is to define a Lagrangian or action L. This is a functional of the various fields obtained by integrating over M a Lagrangian density. For example, for a scalar field theory where the only field is a scalar function φ, the simplest Lagrangian is

$$L(\varphi) = \int_M |\text{grad}\,\varphi|^2\,dx$$

where the norm and volume are those of Minkowski space.

For Yang–Mills theory the Lagrangian is

$$L(A) = \int_M |F_A|^2\,dx$$

where the norm here also uses an invariant metric on G.

Having fixed a Lagrangian $L(\varphi)$ the 'partition function' of the theory (by analogy with statistical mechanics) is the Feynman functional integral

$$Z = \int \exp(iL)\,D\varphi.$$

More generally, for any functional $W(\varphi)$, the unnormalized expectation value of the 'observable' W is defined by the integral

$$\langle W \rangle = \int \exp(iL(\varphi))\,W(\varphi)\,D\varphi.$$

These Feynman integrals are not very well defined mathematically but they can, when used skilfully, be a useful heuristic

† Strictly $d^*F = 0$ is the Yang–Mills equation and $dF = 0$ is the Bianchi identity.

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tool. In particular, perturbation expansions can be computed explicitly.

The Feynman integral provides a relativistically invariant approach. This is its main purpose. In a non-relativistic treatment a quantum field theory is described by a time-evolution operator e^{itH} in a certain Hilbert space $\mathscr{H}$. The infinitesimal generator H is the Hamiltonian of the theory. There are formal rules which, starting from the Lagrangian formulation via the Feynman integral, produce the Hilbert space $\mathscr{H}$ and the Hamiltonian H. The fundamental relation between the two approaches rests on the formula

$$\langle \exp(\mathrm{i}TH)\varphi_0, \varphi_T \rangle = \int \exp(\mathrm{i}L(\varphi))\, \mathrm{D}\varphi$$

where φ_0, φ_T are scalar fields on R^3 (space) and the Feynman integral is taken over all fields $\varphi(x, t)$ which interpolate between $\varphi_0 = \varphi(x, 0)$ and $\varphi_T = \varphi(x, T)$ for $0 \leq t \leq T$. In particular

$$\text{Trace} \exp(\mathrm{i}TH) = \int \exp(\mathrm{i}L(\varphi))\, \mathrm{D}\varphi \qquad (1.2.1)$$

where, in the Feynman integral, φ is a function on $R^3 \times S^1_T$ where S^1_T is the circle of length T.

Witten's version of the Jones theory is defined by a suitable choice of Lagrangian in 2+1 dimensions and this will be described in Chapter 7. Until then we shall be following the non-relativistic Hamiltonian approach, which is mathematically more rigorous.

In gauge theory, classical fields of force are described in terms of curvature. However, gauge theories have global features which can be non-trivial even when all curvatures vanish. This is fundamental for the relations with quantum field theory which are our basic interest. The prototype of this is the Bohm–Aharonov effect in the quantum theory of the electron. This concerns a solenoid with an interior magnetic

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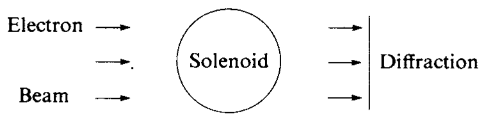
flux but with no external magnetic field. A beam of electrons travelling past the solenoid produces interference patterns indicating a phase-shift. This physical effect takes place even though the electrons travel in a force-free region.



Mathematically the wave-function of the electron in the external region is a section of a flat line-bundle, with non-trivial holonomy round the solenoid.

In non-abelian gauge theories wave-functions are sections of vector bundles and the holonomy lies in a non-abelian group. This is the starting point for the relation between topology and quantum field theory that is embodied in the Jones–Witten theory.

1.3 History of knot theory

The study of knots (and links) in ordinary three-dimensional space is the archetype of a topological problem. Knots are remarkably complicated things and, even with all the sophisticated techniques of modern topology, they have resisted a definitive treatment. The remarkable developments growing out of the Jones polynomial are an indication of the subtlety of knot theory.

A *knot* is by definition a smooth embedding of a circle in R^3. Two knots are equivalent if one knot can be deformed continuously into the other without crossing itself. A *link* is an embedded finite union of disjoint circles.

Knot theory has an interesting history. In the nineteenth century physicists were pondering on the nature of atoms. Lord Kelvin, one of the leading physicists of his time, put

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forward in 1867 the imaginative and ambitious idea that atoms were knotted vortex tubes of ether [32].

The arguments in favour of this idea may be summarized as follows.

(1) *Stability.* The stability of matter might be explained by the stability of knots (i.e. their topological nature).
(2) *Variety.* The variety of chemical elements could be accounted for by the variety of different knots.
(3) *Spectrum.* Vibrational oscillations of the vortex tubes might explain the spectral lines of atoms.

From a modern twentieth-century point of view we could, in retrospect, have added a fourth.

(4) *Transmutation.* The ability of atoms to change into other atoms at high energies could be related to cutting and recombination of knots.

For about 20 years Kelvin's theory of vortex atoms was taken seriously. Maxwell's verdict was that 'it satisfies more of the conditions than any atom hitherto considered'.

Kelvin's collaborator P. G. Tait undertook an extensive study and classification of knots [31]. He enumerated knots in terms of the crossing number of a plane projection and also made some pragmatic discoveries which have since been christened 'Tait's conjectures'. After Kelvin's theory was discarded as an atomic theory the study of knots became an esoteric branch of pure mathematics.

Despite the great strides made by topologists in the twentieth century the Tait conjectures resisted all attempts to prove them until the late 1980s. The new Jones invariants turned out to be powerful enough to dispose of most of the conjectures fairly quickly.

One of the early achievements of modern topology was the discovery in 1928 of the *Alexander polynomial* of a knot or a link [1]. Although it did not help to prove the Tait conjectures

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it was an extremely useful knot invariant and greatly simplified the effective classification of knots. The Alexander polynomial arises from the homology of the infinite cyclic cover of the complement of a knot. Equivalently it can be derived from considering cohomology of the knot complement with coefficients in a flat line-bundle. This is very much the context of the Bohm–Aharonov effect.

For more than 50 years the Alexander polynomial remained the only knot invariant of its kind. It was therefore a great surprise to all the experts when, in 1984, Vaughan Jones discovered another polynomial invariant of knots and links. As already mentioned, this turned out to be extremely useful and enabled several of Tait's conjectures to be established.

In the next section we shall briefly summarize some of the key facts about the Jones polynomials. For an excellent and thorough presentation the reader is referred to the account by Jones in [17].

1.4 The Jones polynomial

The Jones polynomial is a polynomial in t, t^{-1} assigned to a knot K in R^3. It is denoted by $V_K(t)$. It is normalized so that $V(t)=1$ for the unknot (the standard unknotted circle in R^3). Moreover it has the key property

$$V_{K^*}(t) = V_K(t^{-1}) \tag{1.4.1}$$

where K^* is the mirror image of K. Simple examples show that $V_K(t)$ need not be invariant under $t \to t^{-1}$, so that the Jones polynomial can sometimes distinguish knots from their mirror images. For example the right-handed trefoil knot has

$$V(t) = t + t^3 - t^4$$

and so is distinguished from its mirror image. The Alexander polynomial on the other hand always takes the same value for a knot and its mirror image.

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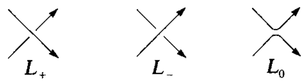
The Jones polynomial can be defined (as a Laurent polynomial in $t^{1/2}$) generally, for any *oriented link L* (i.e. each component of L is oriented). Reversing the orientation of all components leaves the Jones polynomial unchanged. This explains why, for a knot, the orientation is irrelevant.

If we represent a link by a general plane projection with over/under crossings the Jones polynomial can be characterized and computed by a skein relation. Given any oriented link diagram L, and a crossing point, we can alter the crossing to produce three different diagrams as indicated



Let V_+, V_-, V_0 denote the Jones polynomials of these links. Then the skein relation is

$$t^{-1}V_+ - tV_- = (t^{1/2} - t^{-1/2})V_0. \qquad (1.4.2)$$

The skein relation is, in a sense, deceptively simple. There is no obvious reason *a priori* why this relation should define a link invariant: it might depend on the plane presentation.

The way the Jones polynomial was originally discovered was via braids and representations of the Hecke algebra. A braid is a collection of strands as depicted below.

Note that all strands move upwards. Two braids can be composed in an obvious way, giving the braid group on n strands B_n. Formally we can define B_n as the fundamental group of the configuration space C_n of n distinct points in the plane. The usual picture of a braid can then be viewed as

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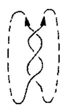
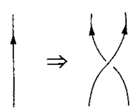
the space–time graph (with time vertical) of motion along a closed path in C_n.

Given a braid β we can form an oriented link $\hat{\beta}$ by closing up the braid in a standard way (see below).



Conjugate elements in B_n give rise to equivalent links. Moreover increasing the number of strands in a braid by a simple twist, as shown,



(1.4.3)

does not affect the corresponding link. A classical theorem of Markov asserts that these two moves generate all equivalences between the resulting links.

Thus to produce an invariant of oriented links one need only produce a *class function* on all B_n which is unchanged by the move (1.4.3).

Since class functions arise naturally as characters of representations this suggests we start by considering representations of the braid groups. In fact Jones used representations which came from the *Hecke algebra* $H(n, q)$. This is the quotient of the group algebra of B_n obtained by requiring the generator σ (a single twist of consecutive strands) to satisfy the quadratic relation

$$(\sigma - q)(\sigma + 1) = 0.$$

If $q = 1$ so that $\sigma^2 = 1$ we get the group algebra of the symmetric group S_n. It follows that, for generic values of q, $H(n, q)$ has the same irreducible representations as S_n.

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For each Young diagram (parametrizing an irreducible representation of S_n) we then get a character of B_n which depends on q (as a Laurent polynomial in $q^{1/2}$). The Jones polynomial (with $t = q$) is a suitable combination of these characters. In fact only the two-rowed Young diagrams are needed.

The Jones polynomial has been generalized in a variety of ways. One way, described in detail in [17], gives a two-variable polynomial. This also satisfies a skein relation and can be constructed from representations of the Hecke algebra, but now using all Young diagrams.

Another, and more fundamental, way involves choosing a compact Lie group G and an irreducible representation. A polynomial invariant of oriented links is then constructed by using solutions of the Yang–Baxter equations. The original Jones polynomial corresponds to taking $G = SU(2)$ with its standard representation on C^2. Taking $G = SU(n)$ for all n, together with their standard representation on C^n, gives polynomials which, taken together, are equivalent to the two-variable polynomial of [17].

Witten's approach, which we shall be describing, also involves a choice of group G and a representation. It produces the relevant polynomials in a more direct and natural manner. Moreover, in Witten's theory, we get invariants for links in arbitrary compact 3-manifolds alone (taking the empty link with no components). This is a major advantage and is a convincing demonstration of the naturality of Witten's method.

It is perhaps worth emphasizing that the algebraic or combinatorial definition of the Jones polynomial is quite elementary and rigorous. It lacks, however, any clear conceptual interpretation. This is precisely what Witten's theory provides, although there are still technical difficulties in developing this side of the theory in all its aspects.

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