The thermomechanics of plasticity and fracture

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Preface

The present book is an outgrowth of my lecture notes for a graduate course on 'Plasticity and fracture' delivered for the past five years to students in Theoretical Mechanics and Applied Mathematics at the Pierre-et-Marie Curie University in Paris. It also corresponds to notes prepared for an intensive course in modern plasticity to be included in a European graduate curriculum in Mechanics. It bears the imprint of a theoretician, but it should be of equal interest to practitioners willing to make an effort on the mathematical side. The prerequisites are standard and include classical (undergraduate) courses in applied analysis and Cartesian tensors, a basic course in continuum mechanics (elasticity and fluid mechanics), and some knowledge of the strength of materials (for exercises with a practical touch), of numerical methods, and of elementary thermodynamics. More sophisticated thermodynamics and elements of convex analysis, needed for a good understanding of the contents of the book, are recalled in Appendices.

The book deals specifically with what has become known as the *mathematical* theory of plasticity and fracture as (unduly) opposed to the *physical* theory of these fields. The first expression is reserved for qualifying the macroscopic, phenomenological approach which proposes equations abstracted from generally accepted experimental facts, studies the adequacy of the consequences drawn from these equations to those facts, cares for the mathematical soundness of these equations (do they have nice properties?), and then, with some confidence, provides useful tools to designers and engineers. The second expression refers to the 'physical' approach which consists in justifying laws governing macroscopic phenomena on the basis of microscopic descriptions at a finer scale (that of the discrete world), and thus is intimately related to solid-state physics, metallurgy and, more generally, what is now called materials science. For example, with such a frame of mind, the slip deformation in crystal plasticity is viewed as resulting from the movement and generation of dislocations, and is generally irreversible due to the atomic potential barriers. Although my own research would incline me to consider this stimulating aspect of plastic phenomena, the present book deals practically exclusively with the macroscopic, smoothed-out scale, physics being used in the guise of irreversible thermodynamics in order to control the macroscopic evolution along the 'Arrow of time'. For the physical theory of plasticity and fracture I recommend Asaro (1983), Cottrell (1953), Honeycombe (1968), Mura (1982), Seeger (1958), Teodosiu (1975, 1982), Zener (1948), and the book of Polukhin *et al.* (1983).

Numerous problems are proposed by way of illustration or to provoke the thoughts of the reader. But the composition of the book is quite different from that of some 'classics' of plasticity theory. Just to quote two examples, while Hill's book of 1950 devotes some seventy pages among its first chapters to a general presentation and general theorems (not many at that time) and then proceeds for several hundred pages to solving problems analytically, including by the slip-line theory (in rigid–plastic bodies), Kachanov's book of 1974, another classic, presents many applications and then concludes with general theorems. Since then the emphasis has shifted, this being faithfully reflected in the theoretical and numerical developments reported in the present book: with the implementation of powerful numerical methods (finite elements, fast integration of systems of evolution equations), analytical solutions have often become outdated while general theorems have gained in importance, perforce. Therefore, most of the present book is rather general in approach with theoretical statements spread all over while some analytical solutions are given because of their everlasting pedagogical value. More on this evolution can be found in Drucker (1988).

Another development reflected in the title is the *thermomechanical* framework which, to the posthumous satisfaction of the great P. Duhem, has been fully integrated into the curriculum in continuum mechanics. This also fully agrees with the late H. Ziegler's point of view (Ziegler, 1963; Ziegler and Wehrli, 1987).

Some inevitable choices had to be made as, originally, the course was delivered in ten sessions of two hours. As a consequence, I think it fair to announce at the outset what the reader will *not* find in the book: non-associated flow rules, non-convex yield surfaces, slip-line theory, stability and wave problems in elastoplasticity, and applications to soil mechanics (except for a very few exceptions). In contrast, the reader *will* find hereinafter elastoplasticity with hardening, an introduction to elastoplasticity in finite strains, homogenization of elastic–plastic composites, coupling between plasticity and damage, path-independent integrals in fracture, numerical treatments of elastoplasticity problems, coupling with thermal fields and singularities, and applications to metals and crystals. That is, all through these pages the plastic potential coincides with the yield surface, which is always supposed to be convex: the normality rule applies and Drucker's 'postulate' holds true (see counterexamples in Houlsby (1981) and Richmond and Spitzig (1980); Mroz (1963), and Ziegler and Wehrli (1987)). Strain-rate dependence is not considered in general, but for comparison purposes. Damage and fracture enter the *same* thermodynamical framework as elastoplasticity. This owes much to the works of Nguyen Quoc Son (in elastoplasticity and fracture) and Pierre Suquet (for composites) from whom we have borrowed much material (especially, Nguyen Quoc Son (1981a) and Suquet (1987)). My admiration for those who have contributed much to this *nonlinear* science is expressed in the list of names given in the following 'Historical perspective', as well as in the Bibliography, obviously far from exhaustive but, in my opinion, doing justice in a balanced manner to North American, West European and East European contributions.

The reader may naturally wander through the pages and chapters (a practice I very much enjoy myself), but if any advice can be given at all the following order of reading may be an astute, if not efficient, one: Chapter 1, then Appendix 1, followed by Chapter 2, completed by Appendix 2, then Chapters 3 and 4 and Appendix 3, and the rest in order of appearance. Chapters 8, 9 and 10 can be skipped in a first reading. In my own practice the contents of the appendices, together with many of the problems, were left to the students as personal reading and homework or term papers.

It is hoped that the book will find its place both on the student's shelf and in the professional scientist's office. I would be most pleased if designers and engineers would consider it as a reference.

My thanks belong to those who have helped me to capture the flavour and peculiarities of elastoplasticity, for me a rather young science. I found efficiency, kindness, encouragement and advice in the CUP team on (Applied) Mathematics and Mechanics. Moreover, the 'home typist', Eleni, the only person who 'took' this course on elastoplasticity without taking first 'Continuum mechanics', has done an excellent job in typing the manuscript. Can I ask for more than that? I feel spoiled.

G.A. Maugin

Historical perspective†

A Landmarks in plasticity theory from 1870 to 1980:

Tresca (1872), Barré de Saint Venant (1871): Flow criterion; torsion problem

Bauschinger (1886): The 'Bauschinger' effect

Mohr (1900): Mohr's circle to express criterion of failure (cited by Timoshenko (1953, p. 286))

Prandtl (1903): Membrane analogy, Prandtl–Reuss relations (Prandtl, 1924; Reuss, 1939)

Huber (1904), Mises R. von (1913): Flow criterion

Hencky (1923): Slip-line theory

Hencky (1924): Hencky's laws

Odqvist (1933): Hardening parameter (Odqvist's parameter)

Melan (1938): Plastic potential

Melan (1938): Shakedown theory (static approach)

Bridgman (1940–1952): High-pressure deformation and flow

Ilyushin (1943): Method of successive elastic solutions

Zener (1948): Rheological models

Hill (1948): Maximal dissipation principle

Hodge, Prager (1949): Minimum principle

Ilyushin (1948): Stability postulate

Greenberg (1949): Minimum principle

Drucker (1951): Drucker's inequality

Symonds (1951): Limit analysis, collapse mechanisms

Koiter (1960): Shakedown theory (kinematic approach)

Ziegler (1963): Plasticity and nonequilibrium thermodynamics, nonlinear generalization of Onsager's relations

Lee (1969): Multiplicative decomposition of finite deformation gradient

Halphen and Nguyen Quoc Son (1975): Notion of generalized standard material

† Full references in Bibliography

Strang, Suquet, Temam (1979, 1980): Functions of bounded variations in elastoplasticity

B Landmarks in fracture theory from 1920 to 1970

Griffith (1920): Surface energy theory
Westergaard (1939): Singularity at crack tip
Irwin and Kries (1951): Elasticity theory of fracture
Eshelby (1951): Path-independent integral (also Cherepanov, 1967, Rice, 1968)
Barenblatt (1960): Elastic theory of cohesive forces at cracks

C Landmarks in damage theory (1858—1963)

Woehler (1858–1870): Cyclic loading, Woehler's curves
Norton (1929): Creep, Norton's law
Kachanov (1958): Notion of damage
Rabotnov (1963): Damage parameter

D Landmarks in related areas (useful for the subject matter of the present book)

Rayleigh, Lord (1873): Dissipation function
Goursat (1898): Biharmonic and analytic functions
Carathéodory (1909, 1925): Axiomatics of thermostatics
Kolossov (1909), Muskhelishvili (1953): Complex-function technique in plane problems
Duhem (1911): The Clausius–Duhem inequality
Sobolev (1936): function spaces of the 'Sobolev' type
Onsager, Casimir, Meixner, de Donder, etc. (1940, 1960): Nonequilibrium thermodynamics (see de Groot and Mazur (1962))
Bridgman (1943): The notion of internal variable
Fenchel (1949): Conjugacy and Legendre–Fenchel transformation (see Rockafellar (1970))
Coleman, Noll, Truesdell (1960–1970): 'Rational thermodynamics' (see Truesdell (1969))
Argyris and others (1965): Finite-element method

Notation

Vectors and Cartesian tensors

Real line: $\mathbb{R}$
Three-dimensional Euclidean space: $\mathbb{E}^3$
Open set of $\mathbb{R}^3$: Ω
Boundary of Ω: $\partial\Omega$
Closure of Ω: $\bar{\Omega}$
Orthonormal basis in $\mathbb{E}^3 : \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$: $\{\mathbf{e}_i\} i = 1, 2, 3$
Kronecker delta (=1 if $i = j$, =0 otherwise): δ_{ij}
Vector: $\mathbf{V} = \sum_{i=1}^{3} V_i \mathbf{e}_i$
Einstein summation convention (on dummy indices): $A_j B_j = \sum_{j=1}^{3} A_j B_j = A_1 B_1 + A_2 B_2 + A_3 B_3$
Second-order tensor $\boldsymbol{\sigma} = \sum_{i,j=1}^{3} \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$
Tensor product $\otimes$: $(\mathbf{A} \otimes \mathbf{B})_{ij} = A_i B_j$
Transposition $\boldsymbol{\sigma}^{\mathrm{T}} = \{\sigma_{ji}\}$ if $\boldsymbol{\sigma} = \{\sigma_{ij}\}$
Symmetric second-order tensor: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ or $\sigma_{ij} = \sigma_{ji}$ (six independent components at most)
Symmetric part: $\boldsymbol{\sigma}_{\mathrm{S}} \equiv \{\sigma_{(ij)} \equiv \frac{1}{2}(\sigma_{ij} + \sigma_{ji})\}$
Anti-(or skew-) symmetric part: $\boldsymbol{\sigma}_{\mathrm{A}} \equiv \{\sigma_{[ij]} = \frac{1}{2}(\sigma_{ij} - \sigma_{ji})\}$
Scalar product of two vectors: $\mathbf{A} \cdot \mathbf{B} = \sum_{j=1}^{3} A_j B_j = A_j B_j$
Vector product of two vectors: $(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k = \varepsilon_{ijk} A_j B_k$

$$\text{Permutation symbol } \varepsilon_{ijk}: \begin{cases} = 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ = -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ = 0 & \text{otherwise} \end{cases}$$

Trace of a tensor $\boldsymbol{\sigma}$: $\mathrm{tr}\,\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_{ii} = \sigma_{jj} = \sigma_{11} + \sigma_{22} + \sigma_{33}$
Deviator of a tensor: $\boldsymbol{\sigma}^{\mathrm{d}} = \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\,\mathbf{1} = \{\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\}$
'Inner' product of two tensors: $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sum_{i,j=1}^{3} \sigma_{ij} \varepsilon_{ji} = \sigma_{ij} \varepsilon_{ji}$.

Differential operators:

Nabla ('del' operator) $\nabla := \{\partial/\partial x_i; i = 1, 2, 3\}$, e.g., $\nabla\phi = \phi_{,i}\mathbf{e}_i$.
Gradient tensor of a vector field: $\nabla\mathbf{A} = \{A_{i,j}\}$

Divergence of a vector $\nabla \cdot \mathbf{A} = \mathrm{tr}(\nabla \mathbf{A}) = A_{i,i} = \sum_{i=1}^{3} A_{i,i}$
Curl of a vector: $(\mathrm{curl}\, \mathbf{A})_i = (\nabla \times \mathbf{A})_i = \varepsilon_{ijk} A_{k,j}$
Normal derivative: $\partial\phi/\partial n = (\mathbf{n} \cdot \nabla)\phi = \phi_{,i} n_i$
Laplacian of a scalar: $\nabla \cdot \nabla\phi = \nabla^2\phi = \phi_{,ii}$

Kinematics, strains:

Position (placement) in $\mathbb{E}^3$: $\mathbf{x}, \mathbf{X}$
Displacement vector: $\mathbf{u} = \{u_i\}$
Linear strain tensor: $\boldsymbol{\varepsilon} = (\nabla \mathbf{u})_S = \{\varepsilon_{ij} = \frac{1}{3}(u_{i,j} + u_{j,i})\}$
Deviator of $\boldsymbol{\varepsilon}$: $\boldsymbol{\varepsilon}^{\mathrm{d}}$
Elastic strain: $\boldsymbol{\varepsilon}^{\mathrm{e}}$
Plastic strain: $\boldsymbol{\varepsilon}^{\mathrm{p}}$
Viscous strain: $\boldsymbol{\varepsilon}^{\mathrm{v}}$
Equivalent strain: $\varepsilon_{\mathrm{p}} = (\frac{2}{3}\varepsilon^{\mathrm{p}}_{ij}\varepsilon^{\mathrm{p}}_{ij})^{1/2}$
Cumulated plastic strain: $\overline{\varepsilon^{\mathrm{p}}} = \int_0^t (\frac{2}{3}\dot{\varepsilon}^{\mathrm{p}}_{ij}\dot{\varepsilon}^{\mathrm{p}}_{ij})^{1/2}\, \mathrm{d}t$
Velocity vector: $\mathbf{v} = \{v_i = \dot{u}_i\}$
Partial time derivative: $\dot{A} = \partial A/\partial t$
Acceleration vector: $\boldsymbol{\gamma} = \dot{\mathbf{v}} = \ddot{\mathbf{u}}$
Strain-rate tensor: $\mathbf{D} = \{D_{ij}\} = (\nabla \mathbf{v})_S$
Rotation-rate tensor: $\boldsymbol{\Omega} = \{\Omega_{ij}\} = (\nabla \mathbf{v})_A$
Vorticity vector: $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$
Finite deformation: $\mathbf{F}$
n-vector of internal variables: α

Forces, stresses:

Stress tensor: $\boldsymbol{\sigma}$
Deviator of stress: $\mathbf{s} = \boldsymbol{\sigma}^{\mathrm{d}}$
Invariants of stresses: $(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}})$ or $I_\alpha(\boldsymbol{\sigma})$, $\alpha = 1, 2, 3$
Mean stress: $\sigma_{\mathrm{m}} = \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}$
Normal stress: σ_{n}
Tangential stress: τ
Pressure: p
Elastic limit: σ_0 (yield stress)
Mises equivalent stress: $\sigma_{\mathrm{Y}} = (\frac{3}{2}\sigma^{\mathrm{d}}_{ij}\sigma^{\mathrm{d}}_{ij})^{1/2}$
Convex of plasticity: C
Plastic multiplier: $\dot{\lambda}$
Viscosity: η_{v}
Loading function: f or F
Body force: $\mathbf{f}$ or $\mathbf{g}$

Surface traction: $\mathbf{T}$
Hardening parameters: α, β
Hardening modulus: h
Force associated to $\boldsymbol{\alpha}$: $\mathbf{A}$
Tensor of elasticity coefficients (rigidities): $\mathbf{E}$
Tensor of elastic compliances: $\mathbf{S}$
Lamé coefficients: λ and μ
Hooke's modulus: E
Poisson's ratio: ν
Damage parameter: D

Thermodynamics:

Thermodynamic temperature: θ
Matter density: ρ
Internal energy per unit mass: e
Entropy per unit mass: η
Free energy per unit mass: ψ
Total internal energy: E
Total kinetic energy: K
Total entropy: $\mathscr{N}$
Total power: $\mathscr{P}$
Total dissipation: Φ
Internal energy per unit volume: $\mathscr{E}$
Entropy per unit volume: S
Free energy per unit volume: W
Source heat per unit mass: h
Heat flux vector $= \{q_i\}$: $\mathbf{q}$
Dissipation per unit volume: ϕ
Viscous dissipation: ϕ_{V}
Plastic dissipation: ϕ_{p}
Thermal dissipation: ϕ_{q}
Intrinsic dissipation: ϕ_{intr}
Dissipation function/potential: $\mathscr{D}$