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155 Classification theories of polarized varieties, TAKAO FUJITA
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Classification Theories of Polarized Varieties

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Contents

Introduction

By a *polarized variety* we mean a pair (V, L) consisting of a projective variety V and an ample line bundle L on it. We will classify such pairs and describe their structure as precisely as possible.

Needless to say, algebraic varieties are the main object in algebraic geometry. In this book, however, we mainly consider the pair (V, L) rather than the variety V itself. There are several reasons of taking this viewpoint.

First of all, polarization (or the linear system defined by it) is very important for describing the structure of a variety. For example, the projective space $\mathbb{P}^n$ is described by a homogeneous coordinate system, namely a linear parametrization of $H^0(\mathbb{P}^n, \mathcal{O}(1))$. But it is by no means easy to recognize a projective space without being given a polarization. For beginners it takes some thought to see that a twisted cubic in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1$, and this is because the polarization $\mathcal{O}(1)$ is not given a priori. Another example is the space parametrizing linear $\mathbb{P}^2$'s contained in a smooth hyperquadric in $\mathbb{P}^5$. This is actually isomorphic to $\mathbb{P}^3$. Is that obvious to you ?

There are polarizations which are not very ample but are useful for this purpose. For example, let $f: V \longrightarrow \mathbb{P}^n$ be a finite double covering. Then $L = f^*\mathcal{O}(1)$ is ample, but not very ample. To recognize V via f is equivalent to considering the pair (V, L), since f is the rational map defined by $f^*|\mathcal{O}(1)|$. In this case the graded algebra

$G(V, L) = \oplus_{t\geq 0}H^0(V, tL)$ has a very simple structure and (V, L) is a hypersurface in a weighted projective space. Although mL is very ample for $m \gg 0$, the graded algebra $G(V, mL)$ is not as simple as $G(V, L)$. Another classical example is Weierstraß's normal form $y^2 = 4x^3 - g_2x - g_3$, which exhibits an elliptic curve as a weighted hypersurface of degree six in the weighted projective space $\mathbb{P}(3, 2, 1)$. In general, by the ampleness of L, V is recovered from $G(V, L)$ by taking Proj. This provides us with an algebraic approach to algebraic varieties. Thus, the study of polarized varieties is indeed an *algebraic* geometry (but of course geometric approaches are also useful).

Second, we note that most interesting examples of algebraic varieties carry natural polarizations. In classical projective geometry the hyperplane section gives such polarizations. In case of Jacobian varieties we have the Θ-divisors. On the other hand, abelian varieties can be recognized by their periods. The interplay of these two approaches yielded the theory of ϑ-functions. We should also observe the importance of polarizations in Torelli type theorems for many types of varieties, e.g. K3-surfaces. In moduli theories you find many examples of moduli spaces which carry polarizations constructed in a natural way. In my opinion, God did not make abstract varieties, but polarized varieties.

Third, we notice the relationship with the theory of singularities. The vertex of a cone over a polarized variety gives a typical example of a singularity. Thus, the category of polarized varieties can be viewed as a subcategory of singularities. Moreover there is an evident similarity of notions and theorems between them. It is not easy to formulate this analogy in a logically strict way, but nevertheless it is useful for heuristic purposes to have this parallelism in mind. Recently classification theory and the theory of minimal models of algebraic varieties has made remarkable progress (cf. [KMM], [Mor5]), where the study of certain singularities (terminal, canonical, log-terminal etc.) plays an important role. I believe that these objects, namely, polarized varieties, singularities and algebraic varieties, should be and will be studied together more and more in the future.

OK, let us now agree that we will study polarized varieties here. But how ?

We want to recognize each polarized variety (V, L) as what it is. However, there are too many types of polarized varieties and there is no almighty method for all of them. We must employ various tools according to the types of (V, L). Therefore we need first to classify polarized varieties, so that we can distinguish those objects for which a certain approach works well. Thus, philosophically, each approach should have its own classification theory. This is why I wrote "*Theories*" in the title of this book. Here I will present two such theories.

The first one may be called Apollonius method. Namely, given a polarized variety (V, L), we take a member D of $|L|$ such that (D, L_D) is also a polarized variety, where L_D is the restriction of L to D. We first study the pair (D, L_D) which is of lower dimension, and then proceed to (V, L) by making use of various results of Lefschetz type. Thus, we use induction on $\dim V$ for proofs. This approach has a long history in classical geometry (this idea can be traced back to Apollonius, so I would like to call this method '*Apollonius method*'). However, since we do not assume that L is very ample, there is not always such a good member D of $|L|$ in general. The theory of Δ-genus, which is defined by $\Delta(V, L) = n + L^n - h^0(V, L)$ where $n = \dim V$, provides several sufficient conditions for the existence of such a member D. Moreover, Δ-genus turns out to be a powerful invariant for characterizing (V, L) if Δ is small enough. Chapter I is devoted to this theory.

In Chapter II we present the theory of adjoint bundles $K + mL$, where K is the canonical bundle of V and m is a positive integer. Here V is assumed to be smooth and defined over $\mathbb{C}$. By a polarized version of Mori-Kawamata's theory on minimal models, we see that (V, L) has a special structure when $K + mL$ is not nef, i.e. $(K + mL)Z < 0$ for some curve Z in V. The description of such a structure is very precise when $n - m$ is small, while $K + (n + 1)L$ is always nef. Using this theory we can classify polarized manifolds by their sectional genus, which can be defined by the formula $2g(V, L) - 2 = (K + (n - 1)L)L^{n-1}$. We remark that adjoint bundles have long been studied (especially when $n = 2$, $m = 1$) by the Apollonius method. Sommese obtained many beautiful results in this way. In our case the assumption is weaker (L is just ample) and we need different techniques, but the conclusions are very similar to classical ones.

In Chapter III we survey Ionescu's classification theory of projective varieties (= varieties *embedded* in projective spaces) and Sommese's theory on adjunction process (when L is ample and spanned), but technical details of the proofs are omitted.

In Chapter IV we discuss several further developments and generalizations, and at the end we present a computer-aided enumeration of ruled polarized surfaces of a fixed sectional genus.

In addition, numerous related topics are mentioned in various parts, which will be useful especially for advanced readers who want to know about the present state of investigations.

There are many interesting and important theories which we do not treat in this book — e.g. moduli theories amongst others. However, I hope that our studies will provide a good starting point for moduli theories in many cases.

The reader is assumed to have some knowledge of algebraic geometry, as can be found in Hartshorne's book [Ha4] for example. For the sake of convenience, in Chapter 0, I give a brief summary of matters which are often used freely in this book. For their proofs the reader should consult appropriate references. This book is not quite self-contained in the sense that we usually present a result with an outline of proof of it, and refer to other papers for technical details. I hope that this rather helps the reader to have a good idea of what is most important. I would be happy if young people can enjoy the experience of applying modern general theories in concrete problems.

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Notation, Convention and Terminology

Basically we employ the customary notation in algebraic geometry as in [**Ha4**].

In most cases we work in the category of algebraic spaces defined over a fixed algebraically closed field $\mathfrak{K}$. They are assumed to be proper and of finite type over $\mathfrak{K}$ unless specifically stated otherwise. By a *variety* we mean an irreducible reduced space. *Manifold* means a non-singular variety. *Point* means a $\mathfrak{K}$-rational point, while *scheme-point* means a scheme theoretical point.

Vector bundles are often identified with the locally free sheaves of their sections, and these words are used interchangeably. Line bundles are identified with linear equivalence classes of Cartier divisors, and their tensor products are denoted additively, while we use multiplicative notation for intersection products in Chow rings.

The pull-back of a line bundle L on X via a morphism $f: Y \longrightarrow X$ is denoted usually by f^*L or L_Y, but sometimes just by L when confusion is impossible or harmless. This convention applies for pull-backs of other objects too. For example, if f is birational and D is a Cartier divisor on X, by D_Y we mean the total transform of D (we *never* mean the strict transform).

To avoid possible confusion, the canonical bundle of a manifold M is denoted by $K(M)$ or K^M, unlike the customary notation K_M. Similarly $\Theta(M)$ and $\Omega(M)$ denote the tangent and cotangent bundles of M respectively. But the normal bundle of a submanifold C in M is denoted by $\mathcal{N}_{C \subset M}$.

Now we list notation used very often.

Sing(X): the set of singular points of a space X.

Supp(Y): the support of a subspace (or subscheme) Y of X.

$\mathcal{O}_X$: the structure sheaf (or the trivial line bundle) of X.

$h^i(X, \mathcal{F})$: the dimension of the i-th cohomology group of a coherent sheaf $\mathcal{F}$ on X.

$\mathcal{F}[L]$ (or $\mathcal{F}(L)$): $= \mathcal{F} \otimes_{\mathcal{O}} L$ for a line bundle L.

$|L|$: the complete linear system associated with L.

$[\Lambda]$: the line bundle associated with a linear system Λ.

BsΛ: the set-theoretic intersection of all the members of Λ

ρ_Λ: the rational map defined by Λ.

$\{Z\}$: the Chow homology class of an algebraic cycle Z.

$\mathcal{E}^\vee$: the dual bundle of a vector bundle $\mathcal{E}$ over X.

$S^k\mathcal{E}$: the k-th symmetric product of $\mathcal{E}$.

$\mathbb{P}(\mathcal{E})$, $\mathbb{P}_X(\mathcal{E})$: the $\mathbb{P}^{r-1}$-bundle associated with $\mathcal{E}$, r = rank $\mathcal{E}$. A point p over x on X corresponds to a linear subspace N_y of codimension one in $E_x \simeq \mathfrak{K}^r$.

$H(\mathcal{E})$: the tautological line bundle on $\mathbb{P}(\mathcal{E})$. The fiber $H(\mathcal{E})_p$ at $p \in \mathbb{P}(\mathcal{E})$ is identified with the quotient space E_x/N_y. $\mathcal{O}[tH(\mathcal{E})]$ is often denoted by $\mathcal{O}(t)$.

H_α, H_β, $\cdots$: the (pull-backs of) $\mathcal{O}(1)$'s of projective spaces $\mathbb{P}_\alpha$, $\mathbb{P}_\beta$, $\cdots$ indicated by the same Greek letters.