Chapter 0.

Summary of Preliminaries

The matters in this chapter are well known among experts and will be freely used in this book. We review them for the convenience of the reader and hope that it serves as a sort of 'dictionary'. The reader is advised to skip this chapter at first, and then to refer to it if necessary. A paragraph $(x.y)$ will be referred to as $(0.x.y)$ in later chapters, but as $(x.y)$ within this chapter.

§1. Relative viewpoint

Many notions and theorems in algebraic geometry can be formulated in a relative situation. Formally this is a very simple (apparently trivial) process, but it provides a powerful approach for many applications. The 'Bible' of this philosophy is **[EGA]**.

(1.1) Let X be an algebraic space defined over an algebraically closed field $\mathfrak{K}$ with $\mathrm{char}(\mathfrak{K}) = p \geq 0$. For any sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of finite type, we can define the relative spectrum $Z = \mathcal{S}pec_X(\mathcal{A})$ with a natural affine morphism $\pi: Z \longrightarrow X$ such that $\pi_*\mathcal{O}_Z = \mathcal{A}$ (cf. [**EGA**;II,§1] or [**Ha4**;II, Ex. 5.17]).

(1.1.1) *Example (Stein Factorization).* Let $f: Y \longrightarrow X$ be a proper morphism of spaces. Then $\mathcal{A} = f_*\mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-algebras, so $\pi: Z \longrightarrow X$ is a finite morphism for $Z = \mathcal{S}pec_X(\mathcal{A})$. Moreover there is a morphism $\Phi: Y \longrightarrow Z$ such that $f = \pi \circ \Phi$ and $\Phi_*\mathcal{O}_Y = \mathcal{O}_Z$. Hence every fiber of Φ is connected. Such a factorization $f = \pi \circ \Phi$ is determined uniquely by f and is called the *Stein factorization* of f.

(1.1.2) *Remark.* If f is surjective, both X and Y are varieties and if Y is normal, then Z is just the normalization of X in the rational function field of Y.

(1.2) Let $\mathcal{G} = \oplus_{d\geq 0} \mathcal{G}_d$ be a sheaf of graded $\mathcal{O}_X$-algebras of finite type over a space X. Then we obtain a space $S = \mathcal{P}roj_X(\mathcal{G})$ with a projective morphism $\pi: S \longrightarrow X$ in a natural way (see [EGA;II,§3] for details). Here we give a few important examples.

(1.3) *Scroll of a vector bundle.*

Let $\mathcal{E}$ be a locally free sheaf of rank r on X and let $\mathcal{G}$ be the symmetric algebra of $\mathcal{E}$. Then $\mathcal{P}roj_X(\mathcal{G})$ is a $\mathbb{P}^{r-1}$-bundle over X, and will be denoted by $\mathbb{P}_X(\mathcal{E})$ or $\mathbb{P}(\mathcal{E})$ from now on. Moreover there is an invertible sheaf $\mathcal{H}$ on $\mathbb{P}(\mathcal{E})$ such that $\pi_*\mathcal{H}^{\otimes t} \simeq S^t\mathcal{E}$ for any $t \in \mathbb{Z}$. This is called the tautological sheaf on $\mathbb{P}(\mathcal{E})$ and is denoted by $\mathcal{O}(1)$. The corresponding line bundle is denoted by $H(\mathcal{E})$ and the pair $(\mathbb{P}(\mathcal{E}), H(\mathcal{E}))$ is called the *scroll* of $\mathcal{E}$.

If $\mathcal{E} = \mathcal{O}_X[E]$ for some vector bundle E, $\mathbb{P}(\mathcal{E})$ is the quotient $(E^\vee - 0(X))/\mathbb{G}_m$ of the total space of the dual bundle $E^\vee$ of E, minus the zero section, modulo the natural $\mathbb{G}_m$-action via the scalar multiplication. Thus each point $y \in \mathbb{P}(\mathcal{E})$ corresponds to a quotient space H_y of $E_{\pi(y)}$ of dimension one. There is a surjection $\pi^*E \longrightarrow H(\mathcal{E})$ of vector bundles on $\mathbb{P}(\mathcal{E})$, and H_y is identified with $H(\mathcal{E})_y$. The kernel of this homomorphism is naturally isomorphic to $\Omega_{\mathbb{P}(\mathcal{E})/X} \otimes H(\mathcal{E})$, where $\Omega_{\mathbb{P}(\mathcal{E})/X}$ is the relative cotangent bundle of π. In particular we have the following canonical bundle formula:

$$K^{\mathbb{P}(\mathcal{E})} = \pi^*(K^X + \det \mathcal{E}) - r\, H(\mathcal{E}).$$

Remark. The word 'scroll' is used in different meanings in several papers.

(1.4) *Fact* (Relative rational map). *Let* $f: Y \longrightarrow X$ *be a morphism,* E *be a vector bundle on* X *and let* L *be a line bundle on* Y. *Suppose that there is a surjection* $f^*E \longrightarrow L$. *Then there is a morphism* $\rho: Y \longrightarrow \mathbb{P}(E)$ *such that* $f = \pi \circ \rho$ *and* $\rho^*H(E) = L$.

For a proof, see [**EGA**;II,(4.2.3)] or [**Ha4**;II,7.12]. L is said to be relatively very ample with respect to f (or f-very-ample for short) if there is a vector bundle E as above such that ρ is a closed embedding. L is said to be f-ample if mL is f-very-ample for some $m > 0$.

(1.5) *Blowing-up* (cf. [**EGA**;II,§8] or [**Ha4**;p.163]).

Let $\mathcal{I}$ be a coherent sheaf of $\mathcal{O}_X$-ideals defining a subscheme C of a variety X. Let $\mathcal{I}^d$ be the d-th power of $\mathcal{I}$ in $\mathcal{O}_X$ and set $\mathcal{S} = \oplus_{d \geq 0}\mathcal{I}^d$, where $\mathcal{I}^0 = \mathcal{O}_X$. Then $X' = \mathcal{P}roj_X(\mathcal{S})$ is called the *blowing-up* of X along C. C is called the *center* of this blowing-up. $\pi: X' \longrightarrow X$ is a birational morphism with the following properties.

(1.5.1) *The inverse image ideal* $\mathcal{I}'$ *on* X', *which is the image of the natural homomorphism* $\pi^*\mathcal{I} \longrightarrow \mathcal{O}_{X'}$, *is a principal ideal defining a Cartier divisor* E *on* X'.

E is called the *exceptional divisor* of π.

(1.5.2) $\pi(E) = C$ *and* $X' - E \simeq X - C$.

(1.5.3) *For any morphism* $f: Y \longrightarrow X$ *such that the image of the homomorphism* $f^*\mathcal{I} \longrightarrow \mathcal{O}_Y$ *is invertible, there is a unique morphism* $g: Y \longrightarrow X'$ *with* $f = \pi \circ g$.

This is called the *universal property* of the blow-up.

(1.5.4) *Let* Z *be a closed subvariety of* X *not in* C

and let $\mathcal{J}_Z$ be the image of $\mathcal{J}$ via $\mathcal{O}_X \longrightarrow \mathcal{O}_Z$. Then the blowing-up Z' of Z with respect to $\mathcal{J}_Z$ is a closed subvariety of X'.

Z' is called the *strict* (or *proper*) *transform* of Z of the blowing-up $X' \longrightarrow X$.

(1.5.5) *If* X *is smooth along* C *and* C *is a submanifold of* X*, then* $(E, \mathcal{O}_E[-E])$ *is isomorphic to the scroll of the conormal bundle of* C *in* X*. The relative canonical bundle* $K^{X'/X}$ *is* $[(r-1)E]$*, where* $r = \operatorname{codim} C$.

(1.6) *Fact* (Blowing-down). *Let* E *be an effective Cartier divisor on* X*. Suppose that* X *is smooth along* E *and that* $(E, [-E]_E)$ *is isomorphic to the scroll of a vector bundle* $\mathcal{N}$ *over a manifold* C*. Then there exists a space* $X^\flat$ *containing* C *as a submanifold such that* X *is the blowing-up of* $X^\flat$ *along* C*, where* E *is identified with the exceptional divisor.*

For a proof, see [Ar] (or [Moi2], [Nkn] if $\mathfrak{K} = \mathbb{C}$). This criterion is not true in the category of schemes.

(1.7) The relative version of the sheaf cohomology is the higher direct image.

For any continuous map $f: X \longrightarrow Y$ of topological spaces and for all $i \geq 0$, we have natural functors $R^i f_*: \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}(Y)$ between the categories of sheaves of abelian groups which have the following properties.

(1.7.0) $R^0 f_* = f_*$.

(1.7.1) *For any exact sequence* $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ *in* $\mathfrak{Ab}(X)$ *and for all* $i \geq 0$*, there are naturally defined homomorphisms* $\delta_i: R^i f_* \mathcal{G} \longrightarrow R^{i+1} f_* \mathcal{E}$ *such that*

$$0 \longrightarrow f_*\mathcal{E} \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{G} \overset{\delta_0}{\longrightarrow} R^1f_*\mathcal{E} \longrightarrow \cdots \longrightarrow R^{i-1}f_*\mathcal{G} \overset{\delta_{i-1}}{\longrightarrow} R^if_*\mathcal{E}$$

$$\longrightarrow R^if_*\mathcal{F} \longrightarrow R^if_*\mathcal{G} \overset{\delta_i}{\longrightarrow} R^{i+1}f_*\mathcal{E} \longrightarrow \cdots$$ is exact in $\mathfrak{Ab}(Y)$.

(1.7.2) *If* Y *is a point, then* $R^if_*\mathcal{F} \simeq H^i(X, \mathcal{F})$.

(1.8) *Fact* (Leray Spectral Sequence). *Let* $f: X \longrightarrow Y$ *and* $g: Y \longrightarrow Z$ *be continuous maps and let* $\mathcal{F} \in \mathfrak{Ab}(X)$. *Then there is a spectral sequence with* $E_2^{p,q} \simeq R^pg_*(R^qf_*\mathcal{F})$ *converging to* $R^{p+q}(g\circ f)_*\mathcal{F}$.

This means that there are objects $E_r^{p,q}$ in $\mathfrak{Ab}(Z)$ for all p, q and $1 \le r \le \infty$ together with homomorphisms $d_r^{p,q}: E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that

1) $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$ and $E_{r+1}^{p,q} \simeq \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1})$ *for any* p, q, $r \in \mathbb{Z}$,

2) $E_2^{p,q} \simeq R^pg_*(R^qf_*\mathcal{F})$,

3) *for each* p, q *there is an integer* $r(p, q)$ *such that* $d_r^{p,q} = 0$, $d_r^{p-r,q+r-1} = 0$ *and* $E_r^{p,q} \simeq E_\infty^{p,q}$ *for any* $r \ge r(p, q)$, *and*

4) *there is a descending filtration of* $R^k(g\circ f)_*\mathcal{F}$ *such that* $R^k(g\circ f)_*\mathcal{F} = F_{0,k} \supset F_{1,k} \supset \cdots \supset F_{\ell,k} = 0$ *for* $\ell \gg 0$ *and* $F_{p,p+q}/F_{p+1,p+q} \simeq E_\infty^{p,q}$.

When $E_r^{p,q} \simeq E_\infty^{p,q}$ for all p and q, we say that the spectral sequence degenerates at the E_r terms.

(1.9) *Fact. If* $f: X \longrightarrow Y$ *is a proper morphism of algebraic spaces and if* $\mathcal{F}$ *is a coherent* $\mathcal{O}_X$*-module, then* $R^if_*\mathcal{F}$ *is coherent on* Y *for every* $i \ge 0$. *Moreover* $(R^if_*\mathcal{F})_y = 0$ *at a point* y *on* Y *if* $\dim f^{-1}(y) < i$.

For a proof, see [**EGA**;III,(4.2.2)].

(1.10) *Fact* (Serre's Vanishing Theorem). *Let* $f\colon X \longrightarrow Y$ *be a proper morphism of algebraic spaces. Then a line bundle* L *on* X *is* f*-ample if and only if, for every coherent sheaf* $\mathcal{F}$ *on* X, *there is an integer* $N(\mathcal{F})$ *such that* $R^i f_*(\mathcal{F} \otimes [tL]) = 0$ *for any* $t \geq N(\mathcal{F})$ *and* $i > 0$.

(1.11) *Fact. Let* $f\colon X \longrightarrow Y$ *be a proper morphism as above and let* $\mathcal{F}$ *be a coherent sheaf on* X. *Suppose that, for every point* x *on* X, *the stalk* $\mathcal{F}_x$ *is a flat* $\mathcal{O}_{Y,f(x)}$*-module via the mapping* $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$. *Then* $\chi(X_y, \mathcal{F}_y) = \sum_i (-1)^i h^i(X_y, \mathcal{F}_y)$ *is a locally constant function in* $y \in Y$, *where* $X_y = f^{-1}(y)$ *and* $\mathcal{F}_y$ *is the restriction of* $\mathcal{F}$ *to* X_y. *Moreover* $h^i(X_y, \mathcal{F}_y)$ *is an upper-semicontinuous function in* y *for each* i. *If this is locally constant at* $y \in Y$, *then* $R^i f_* \mathcal{F} \simeq \mathcal{O}[E]$ *for some vector bundle* E *over a neighborhood of* y, *and* E_y *is identified naturally with* $H^i(X_y, \mathcal{F}_y)$.

For a proof, see [EGA;III,§7] or [Ha4;III,12.9].

(1.12) We have the following criterion for flatness.

Fact (cf. [EGA;IV,(6.1.5)]). *Let* $f\colon X \longrightarrow Y$ *be a proper morphism of algebraic varieties. Suppose that* Y *is smooth and* X *is locally Cohen Macaulay. Then* f *is flat if and only if* $\dim f^{-1}(y) = \dim X - \dim Y$ *for every* $y \in Y$.

§2. Singularities

(2.1) *Fact* (Serre Duality). *Let* M *be a manifold with* $\dim M = n$ *and let* $\omega_M = \Omega_M^n$ *be its canonical sheaf. Then* $\mathrm{Ext}_M^i(\mathcal{F}, \omega_M)$ *is naturally isomorphic to the dual space of* $H^{n-i}(M, \mathcal{F})$ *for every coherent sheaf* $\mathcal{F}$ *on* M.

Here we give a generalization of this fact on singular spaces for the convenience of later use. For a more complete treatment, see e.g. [Ha2].

(2.2) Let X be a closed subspace of a manifold P with $\dim P = N$ (both X and P may be non-complete). Suppose that any irreducible component of X is of the same dimension n. We define $\mathcal{D}_X^q = \mathcal{E}xt_P^{N+q}(\mathcal{O}_X, \omega_P)$. Then

0) $\mathcal{D}_X^q$ *is an invariant of* X *and is independent of the choice of an embedding* $X \subsetneq P$.

1) $\mathcal{D}_X^q = 0$ *unless* $0 \leq -q \leq n$.

2) $\dim \mathrm{Supp}(\mathcal{D}_X^{-q}) \leq q$ *and* $\mathrm{Supp}(\mathcal{D}_X^{-n}) = X$.

3) *When* X *is smooth,* $\mathcal{D}_X^{-n} \simeq \omega_X$ *and* $\mathcal{D}_X^{-q} = 0$ *for* $q \neq n$.

The proofs are easy (see e.g. [F8]).

(2.3) A subvariety Z of X is called an *embedded component* of X if $\dim Z < \dim X$ and if $Z = \mathrm{Supp}(\mathcal{N})$ for some coherent subsheaf $\mathcal{N}$ of $\mathcal{O}_X$. Here, as usual, Supp denotes the set-theoretical support.

Fact (cf. [F8;(1.11)]). *Set* $q = \dim Z$. *Then* Z *is an embedded component of* X *if and only if* $Z \subset \mathrm{Supp}(\mathcal{D}_X^{-q})$.

(2.4) *Definition.* X is said to be *k-Macaulay* at a point x on X if $\dim Z \leq q - k$ for any $0 \leq q < n$ and any component Z of $\mathrm{Supp}(\mathcal{D}_X^{-q})$ with $x \in Z$. This condition means $x \notin \mathrm{Supp}(\mathcal{D}_X^{-q})$ for $q < k$. X is said to be *locally k-Macaulay* if it is so at every point on X.

Then we have the following facts.

1) X *is locally 1-Macaulay if and only if* X *has no embedded component.*

2) *Let* D *be a subscheme of* X *of pure dimension* $n - 1$ *such that* $\mathcal{I}_D = \mathrm{Ker}(\mathcal{O}_X \longrightarrow \mathcal{O}_D)$ *is invertible. Then* D *is*

$(k-1)$-Macaulay at $x \in D$ if X is k-Macaulay at x.

3) *Let D be as above. Then X is k-Macaulay at x if D is k-Macaulay at x.*

These are easily proved. From these facts we deduce further the following.

(2.5) *Fact.* X *is* k*-Macaulay at* x *if and only if* depth$(\mathcal{O}_{X,\mathfrak{p}}) \geq$ Min$(k$, height $\mathfrak{p})$ *for any scheme-point* $\mathfrak{p}$ *with* $x \in \overline{\mathfrak{p}} \subset X$.

This is called usually 'Serre's condition S_k'. Thus, $\mathcal{O}_{X,x}$ is a Cohen Macaulay local ring if and only if X is n-Macaulay at x, namely $(\mathcal{D}_X^{-q})_x = 0$ for $q \neq n$. $\mathcal{D}_X^{-n}$ is the dualizing sheaf in this case.

(2.6) *Fact. Let Λ be a linear system on X and assume that X is k-Macaulay on an open set $U \subset X$ for some $k < n$. Then any general member D of Λ is k-Macaulay on $D \cap U \cap (X - \mathrm{Bs}\Lambda)$.*

This follows from the observation in (2.4).

(2.7) *Fact. Let X be a locally k-Macaulay variety and let H be an ample line bundle on X. Then, for any locally free sheaf $\mathcal{E}$ on X, $H^i(X, \mathcal{E}[-tH]) = 0$ for $i < k$ and $t \gg 0$.*

For the proof, use the spectral sequence with $E_2^{p,q} = H^p(X, \mathcal{D}_X^q \otimes \mathcal{E}^\vee)$ converging to $H^{-p-q}(X, \mathcal{E})^\vee$, which we obtain by using (2.1).

(2.8) *Fact* (Serre's criterion). *A variety V is normal if and only if it is locally 2-Macaulay and its singular locus is of codimension > 1.*

For a proof, see e.g. [EGA;IV,§5.8]. As a corollary we obtain the following.

(2.9) *Fact* (cf. [Sei]). *Let* Λ *be a linear system on a normal variety* V *such that* $\mathrm{Bs}\Lambda = \emptyset$. *Suppose further that* $\mathrm{char}(\mathfrak{K}) = 0$ *or that* Λ *is very ample. Then any connected component of any general member of* Λ *is normal.*

(2.10) Finally we recall a result in Hironaka's desingularization theory [Hirn].

Definition. Let $\pi: X' \longrightarrow X$ be a blowing-up of X as in (1.5). This is said to be admissible if the center C is smooth and X is normally flat along C.

We omit the precise definition of normal flatness. Roughly it means that the algebraic nature of the singularities of X relative to C varies continuously along C.

(2.11) *Fact. Let* V *be a variety and let* S *be a closed subset of* V. *Assume that* $\mathrm{char}(\mathfrak{K}) = 0$. *Then there is a finite sequence* $V' = V_k \longrightarrow V_{k-1} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 = V$ *of admissible blowing-ups such that* V' *is smooth and the inverse image of* S *in* V' *is a divisor whose support has only normal crossing singularities.*

§3. Intersection theories

(3.1) Let $\mathscr{C}$ be the category of algebraic spaces over $\mathfrak{K}$. We use a theory which has the following features:

a) There is a contravariant functor $A^{\cdot}(*)$ from $\mathscr{C}$ to the category of graded rings with unit. Thus, for any morphism $f: X \longrightarrow Y$ in $\mathscr{C}$, we have a ring homomorphism $f^* = A^{\cdot}(f)$: $A^{\cdot}(Y) \longrightarrow A^{\cdot}(X)$ such that $f^*(A^d(Y)) \subset A^d(X)$, $f^*1_Y = 1_X$. The multiplication in $A^{\cdot}(*)$ is called the *cup product*.

b) There is a covariant functor $A_{\bullet}(*)$ from $\mathscr{C}$ to the category of abelian groups. Moreover, for each space X, $A_{\bullet}(X)$ has a natural structure of graded $A^{\bullet}(X)$-module. The action of $A^{\bullet}(X)$ on $A_{\bullet}(X)$ is called the *cap product*.

c) For every morphism $f: X \longrightarrow Y$ in $\mathscr{C}$, the map $f_* \underset{\text{def}}{=} A_{\bullet}(f): A_{\bullet}(X) \longrightarrow A_{\bullet}(Y)$ is a homomorphism of $A^{\bullet}(Y)$-modules, where $A^{\bullet}(Y)$ acts on $A_{\bullet}(X)$ via the map $A^{\bullet}(f)$.

d) $A^d(X) = A_d(X) = 0$ unless $0 \leq d \leq \dim X$. Moreover, if Y is a point, there is an isomorphism $\deg: A_0(Y) \simeq \mathbb{Z}$. So we have $\deg_X = \deg \circ A_0(f): A_0(X) \longrightarrow \mathbb{Z}$, where $f: X \longrightarrow Y$ is the trivial map.

e) For each complete variety V with $n = \dim V$, there is a natural element $\{V\}$ in $A_n(V)$. Moreover, for any morphism $f: V \longrightarrow W$ of varieties of the same dimension n, we have $f_*\{V\} = d\{W\}$ for the mapping degree d of f.

f) If V is a smooth complete variety with $n = \dim V$, the morphism $\delta: A^d(V) \longrightarrow A_{n-d}(V)$ defined by the cap product with $\{V\}$ is bijective for any d. $\alpha \in A^d(V)$ and $\delta(\alpha)$ are said to be *Poincaré duals* of each other. A subvariety Z is said to be Poincaré dual of α if $\iota\{Z\} = \delta(\alpha)$ for the inclusion $\iota: Z \subsetneq V$.

(3.2) *Example*. When $\mathfrak{K} = \mathbb{C}$, by setting $A^d(X) = H^{2d}(X_{an}; \mathbb{Z})$ and $A_d(X) = H_{2d}(X_{an}; \mathbb{Z})$, we obtain a theory with the above property.

(3.3) There exists a theory as in (3.1) such that $A_d(X)$ is the group of algebraic cycles of dimension d on X modulo rational equivalence (see [Ful]). In this case $A^{\bullet}(X)$ is called the *Chow ring* of X.