CHAPTER 1

Auction theory

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1 Introduction

Auctions are one of the oldest surviving classes of economic institutions. The first historical record of an auction is usually attributed to Herodotus, who reported a custom in Babylonia in which men bid for women to wed.[1] Other observers have reported auctions throughout the ancient world – in Babylonia, Greece, the Roman Empire, China, and Japan.[2]

As impressive as the historical longevity of auctions is the remarkable range of situations in which they are currently used. There are auctions for livestock, a commodity for which many close substitutes are available. There are also auctions for rare and unusual items like large diamonds, works of art, and other collectibles. Durables (e.g., used machinery), perishables (e.g., fresh fish), financial assets (e.g., U.S. Treasury bills), and supply and construction contracts are all commonly bought or sold at auction. The auction sales of unique items have suggested to some that auctions are a good vehicle for monopolists. But it is not only those in a strong market position who use auctions. There are also auction sales of the land, equipment, and supplies of bankrupt firms and farms. These show that auctions are used by sellers who are desperate for cash and willing to sell even at prices far below replacement cost.

The first draft of this paper was written while I was a Fellow at the Institute for Advanced Studies of the Hebrew University of Jerusalem. Discussions with Charles Wilson, Motty Perry, and, especially, Ariel Rubinstein contributed enormously to my understanding of the relation between auctions and bargaining. Comments by Byung-Il Choi and Alvin Roth on a previous version of this manuscript led to improvements in the exposition.

[1] Herodotus may not have been the first to publish. Some scholars interpret the biblical account of the sale of Joseph (the great-grandson of Abraham) into slavery as being an auction sale.

[2] For a more detailed history of auctions and a description of some of the auctions used in the modern world, see Cassady (1967).

Indeed, the only clear common denominator for the kinds of objects that are sold at auction is the need to establish individual prices for each item sold. Used cars, whose condition varies over a wide range, are sold to dealers at auction; new cars are not. Livestock are sold at auction even though close substitutes are readily available, because individual animals differ in weight and health. The price of fresh fish needs to be determined daily, because the daily supply of fish varies so tremendously. Construction contracts are normally too complex to allow a simple pricing schedule to work; competitive bids sometimes provide a workable alternative.

In this essay, I review only a small part of auction theory – the part that claims to explain the long and widespread use of auctions and competitive bidding and to account for certain details of the way auctions are usually conducted. These details include the popular use of sealed-bid and ascending-bid auctions, the establishment of minimum prices, the preparation of expert appraisals of items being sold, and so forth.

Logically prior to explaining the use of auctions is defining just what an auction is. The characteristic feature of an auction is that there is an explicit *comparison* made among bids. In the ascending-bid ("English") auction, a bidder's offer remains open long enough for other bidders to make counteroffers, so that the seller can take the highest offer. In the sealed-bid auction, the bidders' offers are all made simultaneously, so that the seller can compare them directly. In the descending-bid ("Dutch") auction, the seller makes a series of price offers, declining over time. Each bidder has the opportunity to accept or reject the seller's latest price offer; this affords the seller an opportunity to compare the timing of buyers' offers, and to take the offer that is made earliest. Each of these auctions requires that all the bidding be completed within a relatively short period of time. They can be contrasted with, say, a sequential bargaining process in which the seller negotiates one-by-one with a series of buyers who make short-lived offers, so that the seller has no opportunity to compare the simultaneous offers of competing buyers. We shall develop the importance of this difference in more detail later.

The simplest explanation of the continuing popularity of auctions is that auctions often lead to outcomes that are efficient and stable. More formally, in a static deterministic model, the set of perfect equilibrium trading outcomes obtained in an auction game (as the minimum bid is varied) coincides with the set of core allocations. An outcome is in the *core* when there is no coalition of traders that can, by trading just among its members, make all coalition members better off.

To understand the significance of this conclusion, imagine a situation in which a single item is sold but the resulting allocation lies outside the core. There are two possibilities. First, the allocation may be inefficient;

in this case, the new owner will likely find it profitable to resell the item to a buyer who values it more. The second possibility is that, even though the allocation is efficient, there are other buyers around who were willing to pay a higher price (and after the auction are willing to tell the seller so). In either case, the seller may well resolve not to be so quick to sell the next time around and perhaps even to compare alternative offers – that is, to conduct some kind of auction.

A second explanation of the popularity of auctions highlights the advantages of an auction to a seller in a relatively poor bargaining position[3] (such as the owner of a nearly bankrupt firm) when the goods sold at auction can later be resold. Consider the problem of such a seller. Suppose that there are two potential buyers: Mr. 1, who has a high valuation for the item being sold, and Mr. 2, whose valuation is lower. What happens if the seller conducts an auction with a low minimum price? At the equilibrium of the auction game, the item will be sold to Mr. 1 for approximately its value to Mr. 2. With the possibility of resale, that value cannot be less than the price that Mr. 2 could get by reselling to Mr. 1. By conducting an auction, the seller expects to get about the same price as Mr. 2 would get, even though Mr. 2 may be much better positioned for face-to-face bargaining with Mr. 1. Thus, a seller in a relatively weak bargaining position can do as well as a strong bargainer by conducting an auction.

These first two explanations of the prevalence of auctions are developed in detail in Section 2, which focuses on deterministic auction models. A third explanation, reviewed in Section 3, is that even a seller in a strong bargaining position will sometimes find it optimal to conduct an auction. That is, the seller will prefer to conduct some standard auction, such as the sealed-bid or ascending-bid auction with a suitably chosen minimum price, rather than to play any other exchange game[4] with the bidders.

The three explanations just described are, of course, complementary. Together, they provide a cogent set of reasons for a seller to use an auction when selling an indivisible object over a wide range of circumstances.

In the auction models discussed so far, there is little that can be said about the details of how auctions are conducted. In those models, many kinds of auctions (including all the usual ones) lead to the same mean price. However, this "independence" result depends on the assumption that bidders have no private information about each other. Formally, the observations they make are assumed to be statistically independent. When

[3] That is, a poor bargaining position relative to the potential buyers.

[4] An exchange game is any game whose outcome determines an allocation and time of trade, and in which each player has a strategy of nonparticipation that leaves him with his initial allocation.

there is correlated uncertainty on the part of the bidders, different auction rules lead to different mean prices.

In Section 4, we introduce correlated uncertainty into the bidding model and focus on the strategies open even to a seller with no bargaining power, that is, one who cannot commit himself to withhold an item that attracts only low bids.[5] What strategies can such a seller adopt? For one, he can normally choose which kind of auction to offer, provided the minimum bid is kept low, because buyers will always want to participate in the auction.[6] Normally, the seller can also decide whether to reveal any information about the item being sold or about the potential buyers, because it always pays a buyer to listen if he can do so without being seen. Given these options, the seller's preferences are surprisingly systematic. In a wide range of circumstances,[7] the seller will prefer (1) to conduct an ascending-bid auction rather than a sealed-bid auction, (2) to reveal all information that he has available, and (3) to link the price to any available exogenous indicators of value.

The analysis leading to these conclusions is founded on what has been called the *Linkage Principle.* Intuitively, a bidder's expected profits from an auction are greatest when he has private information that the item being sold is quite valuable. The intuition of the Linkage Principle is that the auctions yielding the highest average prices are those that are most effective at undermining the privacy of the winning bidder's information, thereby transfering some profits from the bidders to the seller. According to the principle, privacy is undermined by linking price to information other than (but correlated with) the winning bidder's private information.

The three conclusions described above all follow from the Linkage Principle. In an ascending-bid auction, the equilibrium price depends on the information of losing bidders through the bids they place. That dependence, or linkage, is absent in the sealed-bid auction. Its presence in the ascending-bid auction leads to a higher predicted price (provided that the bidders' information is correlated).

[5] In Section 4, we review some game-theoretic arguments supporting the presumption that a "rational" seller cannot hold out for a high price when he is uncertain about the buyers' reservation prices.

[6] No matter what strategies the other players adopt, each buyer does at least as well by entering the minimum bid as by abstaining from the auction. For some strategies – namely, when others refrain from bidding – he does better. (This argument is transparent for the case where resale is impossible, and can be extended also to the case with resale possibilities.)

[7] The principal assumptions required include risk neutrality, symmetric uncertainty about the bidders' valuations, and a strong form of nonnegative correlation, known as *affiliation,* among the bidders' valuations.

In any kind of auction, the seller, by revealing information, influences the bids and therefore the price. So, by revealing his information, the seller links the price directly to his information. Thus, according to the Linkage Principle, a policy of revealing information raises the expected price that will result from the auction, provided that the information to be revealed is affiliated[8] with the bidders' information. Similarly, basing the price in part on ex post indicators of value creates a linkage that on average increases the expected price (if these indicators are affiliated with the bidders' information). Examples of contracts let at auction where price is determined in part by ex post indicators include construction contracts with a cost-sharing provision and petroleum drilling contracts that provide for royalty payments based on actual production.

The main theme of explaining the prevalence and robustness of auctions is continued in Section 5, where the possibility of collusion is brieffy studied. Collusion is widespread in real auctions, and there is little a one-time seller can do to prevent it when the bidders have a long-term relationship. However, it is shown that ascending-bid auctions are more vulnerable to collusive agreements among bidders in a long-term relationship than are sealed-bid auctions. This is an important reason for industrial firms to solicit sealed bids from suppliers, despite the general superiority of ascending-bid auctions in one-shot competitive situations.

2 Auctions, bargaining, and the core

We begin by formulating and proving the claim that the trading outcomes of the auction game coincide with the core of the corresponding exchange game. This result provides a simple, partial answer to the question of why auction institutions are so prevalent throughout the world and throughout history.

Consider a deterministic setting with a single seller and n (potential) buyers for some item. Let s be the monetary value of the item to the seller; this means that if the seller had the option of selling for some price p or not selling the item at all, he would choose to sell for p if and only if $p \geq s$. Similarly, the buyers have monetary valuations $b_1, \ldots, b_n$. Our model is discrete: All the valuations and bids are multiples of some common unit. Here and throughout this chapter, we make the standard game-theoretic assumption that the deterministic parameters are common knowledge

[8] Random variables are said to be *affiliated* when they are positively correlated conditional on lying in any small rectangle. For example, any pair of positively correlated joint normal random variables are affiliated. A precise formal definition of the concept is given in Section 4.

among the buyers and the seller.[9] Without significant loss of generality, we may assume that $b_1 > \cdots > b_n$ and limit attention to the case where there are some potential gains from trade: $b_1 > s$.

Now, if the seller offers the item for sale using a sealed-bid auction with minimum price $m < b_2$,[10] what will happen? Using any sensible equilibrium concept (e.g., Nash equilibrium in undominated strategies,[11] perfect equilibrium, "rationalizable" strategies, or even correlated equilibrium), the item will be sold to bidder 1 for his bid of b_2.[12] The same trade will occur if the seller sets any minimum price not exceeding b_2. Again, the same will occur if the seller hires an auctioneer to conduct an ascending-bid auction, regardless of whether the bids are called by the bidders themselves or at a slow pace by the auctioneer.

If the seller sets a minimum price $m \in (b_2, b_1)$, the equilibrium outcome assigns the item to bidder 1 for a price of m. Of course, if $m > b_1$,

[9] In the standard theory of games, the players need to know this structure in order to compute the equilibrium and determine how to play. An alternative view, relevant to auction theory, holds that players learn from experience about the reduced form of their decision problems and select their best bids for that problem. Equilibrium is then a state where all players have correctly learned and are using optimal strategies in their decision problems. Mathematically, this leads to the same definition of equilibrium as does the standard view, but it raises different stability questions and does not require as much knowledge among the players about the overall structure of the game.

[10] We assume in this auction and all those considered hereafter that ties are broken by tossing a fair coin.

[11] Although the Nash equilibrium and its refinements are often justifiably criticized, they are particularly well suited to the analysis of auction games. A Nash equilibrium can be defined as a profile of strategies, one for each player, such that (1) each player is maximizing given his beliefs about how the others will play and (2) those beliefs are correct. The first condition is neither stronger nor weaker than the usual rationality assumption in economic models. The second ("rational expectations") condition is most plausible for institutions – such as auctions – that have existed for millennia and so for which expectations can be based on actual experience.

[12] Any perfect equilibrium (Selten, 1975) is a Nash equilibrium in undominated strategies, and in fact for this game the two concepts coincide. In the two-bidder game, the set of perfect equilibria are characterized as follows: Bidder 1 bids b_2. Bidder 2 uses any mixed strategy F that satisfies two conditions. First, $F(b_2^-) = 1$. Second, let $G(x) = [F(x) + F(x-1)]/2$; then $G(x) \le (b_1 - b_2)/(b_1 - x)$ for all $x \in (m, b_2)$. With more than two bidders, one can specify the strategies of the others arbitrarily, provided bidder j always bids less than b_j, and this remains a perfect equilibrium.

Rationalizable strategies are derived by eliminating weakly dominated strategies from the strategy set to form a reduced game. Then weakly dominated strategies are eliminated from the reduced game, and so on until the process ends. The strategies that survive are called *rationalizable*. The only such strategies for bidders 1 and 2 are to bid b_2 and $b_2 - 1$, respectively.

Correlated equilibria (Aumann, 1973) of bidding games employ only rationalizable strategies, so that concept is covered also.

no exchange takes place; in that case the seller's payoff is s and each buyer's payoff is zero. The case $m=b_1$ is somewhat degenerate; its equilibria include both the no-trade outcome and a trade at price b_1. Our earlier choice of the phrase "equilibrium trading outcomes" was intended to denote all the equilibrium outcomes except the no-trade outcome. Our claim is then justified by the following proposition.

Proposition 1. *The set of perfect equilibrium outcomes of the auction game, as the minimum price ranges from s to b_1, consists of the core outcomes of the corresponding exchange game together with the no-trade outcome. The latter can only occur when the minimum price is b_1.*

Proof: Let $x=(x_0, x_1, \ldots, x_n)$ be the vector of payoffs that are received by the seller and the n buyers, respectively. A vector of payoffs x is called an *imputation* if it is individually rational (i.e., nonnegative) and Pareto optimal and corresponds to some feasible allocation of the goods and money among the players. These imply:

$$x_0+x_1+\cdots+x_n=b_1. \tag{2.1}$$

To be in the core, an imputation must also satisfy inequalities asserting that no coalition could, by agreeing to exchange among themselves, earn a higher total payoff:

$$x_0+\sum_{i\in S} x_i \geq \max\{s, (b_i;\ i\in S)\}, \quad \text{for all } S\subset\{1,\ldots,n\}. \tag{2.2}$$

In view of the preceding discussion, the proposition asserts that the core consists entirely of points of the form

$$(x_0, b_1-x_0, 0, \ldots, 0) \quad \text{for } \max(s, b_2)\leq x_0\leq b_1.$$

It is easy to check that all such points satisfy (2.1) and the inequalities (2.2), and so in fact do lie in the core.

Conversely, suppose x lies in the core. From (2.1) and nonnegativity, $x_0+x_1\leq b_1$. From (2.2) for $S=\{1\}$, $x_0+x_1\geq b_1$. Hence, $x_0+x_1=b_1$ and, by (2.1) and nonnegativity, $x_2=\cdots=x_n=0$. Therefore, all points in the core are of the form $(x_0, b_1-x_0, 0, \ldots, 0)$. Using (2.2) with $S=\{2\}$, one finds $x_0+x_2\geq\max(s, b_2)$; so $x_0\geq\max(s, b_2)$. Nonnegativity of x_1 implies $x_0\leq b_1$. ■

The strategic equivalence of the Dutch and sealed-bid auctions and the notion of perfect equilibrium do not transfer neatly to bidding games with continuous bid spaces. For discrete bid spaces with bid increment ϵ, the only perfect equilibrium in the Dutch auction is for the highest evaluator to stop the auction when the price reaches b_2, and for each other player i to stop it at the price $b_i - \epsilon$. There are no corresponding strategies in the standard formulation of the continuous Dutch auction, because there is no possibility of bidding b_i "minus an infinitesimal." Indeed, in the standard formulation of the continuous Dutch auction, no subgame perfect equilibrium exists.

To avoid this problem, we formulate the extensive form Dutch auction game so that a bidder can claim the object whenever the price falls to p, which we call bidding p, or whenever the price falls strictly below p, which we call bidding p^-. If a player bids p, another bids p^-, and all others bid less, then the item is awarded to the one who bids p for price p. If a player bids p^- and nobody else bids more, then the item is awarded to that bidder for a price of p. This specifies a well-defined continuous Dutch auction game which suitably generalizes the game with discrete bid amounts. Moreover, like the discrete bids game, it does have a unique subgame perfect equilibrium: Player 1 bids b_2 and each $i \neq 1$ bids b_i^-.[13]

There still remains the problem that "trembling-hand" perfect equilibrium is undefined for sealed-bid auction games with a continuum of possible bids. To avoid unnecessary technical difficulties, we shall normally limit our analysis to equilibria of Dutch auctions.

From the perspective of cooperative game theory, the seller's ability to set any particular minimum price and stick to it measures his bargaining power.[14] Indeed, the case $n = 1$ is just a bargaining problem, and auction theory predicts (as does core theory) only that the outcome will be efficient and that nobody will be worse off at equilibrium than if they did not trade. Evidently, a complete auction theory must be informed to some degree by bargaining theory. This leaves open the possibility that the predictions of auction theory could be quite sensitive to the bargaining model used.

Actually, when there are several viable bidders, auction theory is surprisingly insensitive to the bargaining theory used at its foundations. To show this, we embed the auction model in a general discounted, infinite-horizon, noncooperative model of bargaining in which an owner always

[13] One could, of course, define a modified sealed-bid auction game that is strategically equivalent to our continuous Dutch auction game. However, comparing the subgame perfect equilibria of the Dutch auction game (identified in the text) with the trembling-hand perfect equilibria of the corresponding sealed-bid auction (identified in note 12) shows that the two games are not equivalent for the purposes of perfect equilibrium analysis.

[14] The role of commitment in bargaining has been analyzed by Crawford (1982). The associated roles of patience and risk aversion have been given a particularly penetrating analysis by Binmore, Rubinstein, and Wolinsky (1986).

has the right to resell anything he has bought. Because a single player may be sometimes a buyer and sometimes a seller, we shall designate a player's valuation by v_i rather than by s or b_i. It is assumed that $v_1 > \cdots > v_n > 0$.

Let Γ^i be a game form that is to be played when i is the owner of the durable good. Thus, $\Gamma^i = (\{\Sigma_j^i;\ j=1,\dots,n\},\ f^i)$, where Σ_j^i is the set of strategies available to j in the game form and f^i is a function mapping strategy profiles into outcomes. Time is modeled as discrete. An outcome involving trade specifies a date of trade $t \geq 1$, a (nonnegative) price p, and the next owner j. There is also an outcome called "no trade" that we identify as a trade at date $t = \infty$. To interpret the results that follow, it will be useful to think of t as the period of i's ownership, rather than to associate t with any actual date.

Certain specified strategies are assumed to be available to the players in each game form Γ^i. First, the owner is permitted to keep the item for himself; that is, he may choose a strategy that always leads to no trade. Second, the owner is permitted to offer a Dutch auction with a zero minimum price. Such an offer, if made, is the first move in Γ^i and initiates an auction subgame (actually, a "subgame form"). If any non-owner bids in the auction, Γ^i ends at date 1 with the item being assigned according to the usual Dutch auction rules. Non-owners must decide simultaneously whether to bid. If no bids are made, play continues according to the continuation rules of Γ^i, whatever they may be. Each non-owner is assumed to have a strategy of refusing to be party to any trade, in which case no payment can be required of him. The assumption that the decision of whether to offer an auction immediately is the first move in Γ^i means that non-owners have no way, before an auction is offered, to commit themselves not to trade.

Using these very general game forms, which specify the rules governing trade given the owner's identity, we create a game in which the buyer can (if he chooses) resell the good. Let player i_0 be the initial owner. Then the game form Γ^{i_0} is played. If the outcome involves trade after a period of ownership of length t_0, at price p_0, and with next owner i_1, we continue with game form Γ^{i_1}, which determines a period of ownership t_1, price p_1, and next owner i_2. The outcome of this sequence of trades specifies that i_0 owns the item from date 0 to date $t_0 - 1$, i_1 owns the item from t_0 to $t_0 + t_1 - 1$, and generally i_j owns it from date $t_0 + \cdots + t_{j-1}$ to date $t_0 + \cdots + t_j - 1$. Payments are made on the dates of transfer of ownership. The number of actual times the good changes hands can be finite or infinite.

The payoff associated with any outcome for any fixed player j is the present value of the flow of benefits he receives plus the net present value of payments received minus payments made. To make this more precise, fix an outcome path. Let $1_j(t)$ be one if player j owns the item on date t and zero otherwise. (In particular, $1_j(-1)=0$.) Let $p(t)$ be the price paid in any trade at date t, or zero if there is no trade at t. Then, j's payoff in the game is:

$$\sum_{t=0}^{\infty} \delta^t[(1-\delta)v_j 1_j(t)+p(t)[1_j(t-1)-1_j(t)]]. \tag{2.3}$$

Thus, δ is the discount factor for the players' payoffs.

With this, the specification of the selling games is completed. Corresponding to each player i there is a game in which the identity i_0 of the initial owner is i. We shall call that game Γ_i^*.

The games Γ_i^* that can be constructed in this way for some choice of game forms Γ^i form a huge class. Included are games where the seller can conduct auctions with a positive minimum price, exclude some set of bidders, bargain effectively with some buyers, commit himself to take-it-or-leave-it offers, or do all of these. Indeed, the only important restrictions on the set of options available to a seller are that he can neither compel a non-owner to buy nor prevent a buyer from reselling the good, and that he can always offer an auction with a zero minimum price. An additional "stationarity" restriction will be imposed through the equilibria that we isolate for study.

In general, a strategy for a player specifies how to play at each date as a function of the date and the entire past history. For our analysis, we limit attention to equilibria in which the players adopt stationary strategies. A *stationary strategy* for player j is an n-tuple $\sigma_j=(\sigma_j^1,\dots,\sigma_j^n)$ such that $\sigma_j^i\in\Sigma_j^i$. Such a strategy specifies how player j should play in each game form Γ^i (he should play σ_j^i) without regard to the earlier history of play. By a *stationary perfect equilibrium,* we mean an n-tuple of stationary strategies $(\sigma_1,\dots,\sigma_n)$ that is a perfect equilibrium profile regardless of the identity of the initial owner (i.e., in each of the games Γ_i^*).

Given a strategy profile $(\sigma_1,\dots,\sigma_n)$, one can define for each player i a value v_i^* associated with owning the item, that is, with playing the game Γ_i^*. With stationary strategies, v_i^* is also the continuation payoff or value of acquiring ownership at any point in the game, regardless of the previous history of play. With nonstationary strategies, that value might depend on the history of play, because future play could also depend on the history.

> ***Proposition 2.*** *Assume there are least three players, $n\ge 3$. Let v_i^* be the expected payoff to i at a stationary perfect equilibrium in the game Γ_i^*. Then $v_1^*=v_1>v_i^*$ for all $i\neq 1$. Let a_i^* be the payoff to i in Γ_i^* if all players except i adhere to their equilibrium strategies while i deviates to adopt a strategy that entails conducting*