1. Exceptional Collections, Mutations and Helices

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We shall give a general axiomatic presentation of the theory of helices and introduce some general definitions and notations.

Research on exceptional bundles was started in Moscow University after a lecture given by A.N. Tyurin given in the autumn of 1984 on a preprint of [**1**]. In that paper a theorem is given describing the possible Chern classes which a stable bundle on $\mathbf{P}^2$ can have. Exceptional bundles appeared as some sort of boundary points. The results of the first one and a half years of our work were presented in [**4**]. Papers [**3**] and [**6**] together with subsequent papers represent the research of the following one and a half years. Most of the papers use a technique which should be called Helix Theory.

The definition of a helix and the first results about helices appeared in [**4**]. The key lemma 2.2 of that paper and the first version of the definition of a helix are due to Gorodentsev [**3**]. These constructions were a generalisation to arbitrary dimensions of a method of Rudakov which assigned an exceptional bundle on a projective plane to a pair of exact sequences [**5**]. The word "helix" and the idea of considering a helix as an infinite system of bundles with some form of periodicity is due to W.N. Danilov.

Further development of the notion of a helix was connected with applications of the primary ideas in new contexts. This was done by Gorodentsev for arbitrary categories of coherent sheaves [**3**], by Rudakov for a category of symmetric sheaves on a two-dimensional quadric [**6**] and by others in subsequent papers. The aim of the present paper is to formulate the most general definition of a helix, which will encompass all of the present applications.

1. Helices.

We shall consider pairs of objects of a category $\mathcal{U}$ or elements of a set $\mathcal{U}$.

In both cases we need to distinguish certain pairs. A pair (A, B) is *left admissible* if a certain pair $(L_A B, A)$ is defined and is *right admissible* if the pair $(B, R_B A)$ is defined.

(1L) If (A, B) is left admissible then $(L_A B, A)$ is right admissible and $R_A L_A B = B$.

(1R) If (A, B) is right admissible then $(B, R_B A)$ is left admissible and $L_B R_B A = A$.

If (1L) and (1R) hold then there is a one-one correspondence between left and right admissible pairs, but it does not eliminate the possibility that some pairs may be both right and left admissible.

The pair $(L_A B, A)$ is called the *left mutation* of (A, B), and $(B, R_B A)$ the *right mutation*. We will also call the object $L_A B$ the *left shift* of B, and $R_B A$ the *right shift* of A.

(2L) Let A, B, C be such that the pairs (B, C), $(A, L_B C)$ and (A, B) are left admissible. Then the pairs (A, C), $(B', L_A C)$ are left admissible, where $B' = L_A B$, and $L_A L_B C = L_{B'} L_A C$.

(2R) Let A, B, C be such that the pairs (B, C), $(R_B A, C)$ and (A, B) are left admissible. Then the pairs (A, C), $(R_C A, B')$ are right admissible, where $B' = R_C B$, and $R_C R_B A = R_{B'} R_C A$.

Note that if axioms (1R) and (1L) are satisfied then condition (2L) is equivalent to the following:

(2L) Let A, B, C be such that the pairs (A, B), $(R_B A, C)$ are right admissible and the pair (B, C) is left admissible. Then the pairs (A, C'), $(R_{C'} A, B)$ are right admissible, where $C' = L_B C$, and $R_C R_B A = R_B R_{C'} A$.

It is easy to formulate an analogous condition for (2R). The equalities in Axioms (2L) and (2R) are usually called the *triangle equations*.

It will be convenient to denote the object $L_A L_B C$ which appeared in (2L) by $L^{(2)} C$ and also to set $R^{(2)} A = R_C R_B A$. In the same way, if $A_1, \dots, A_s$ is a system of objects we put $L^{(0)} A_s = A_s$, $L^{(1)} A_s = L_{A_{s-1}} A_s$, $L^{(i)} A_s = L_{A_{s-i}} L^{(i-1)} A_s$, with the condition that the resulting pairs are left admissible. Analogous notation will be used for right mutations.

DEFINITION. The collection $\{A_i \mid i \in \mathbb{Z}\}$ will be called a *helix of period n* if for all s the following condition is satisfied:

(hel) The pairs (A_{s-1}, A_s), $(A_{s-2}, L^{(1)} A_s), \dots, (A_{s-n+1}, L^{(n-2)} A_s)$ are left admissible and $L^{(n-1)} A_s = A_{s-n}$.

Further on we shall be assuming that (1L), (1R), (2L) and (2R) are satisfied. Then (hel) is equivalent to

(hel)$'$ The pairs (A_{s-n}, A_{s-n+1}), $(R^{(1)} A_{s-n}, A_{s-n+2}), \dots, (R^{(n-2)} A_{s-n}, A_s)$ are right admissible, and $R^{(n-1)} A_{s-n} = A_s$.

It often happens that there is an involution $*$ on $\mathcal{U}$; $A^{**} = A$. We will say that the involution $*$ is compatible with mutations if (A, B) is left admissible precisely when (A^*, B^*) is right admissible, and $(L_A B)^* = R_{A^*} B^*$. Given such an involution, theorems concerning left mutations will imply analogous theorems for right mutations and vice-versa.

Each collection of the form $A_i, A_{i+1}, \dots, A_{i+n}$ is called a *foundation* of the helix $\{A_i\}$. Note that a helix is uniquely determined by any of its foundations.

A collection $\{B_i,\ i \in \mathbf{Z}\}$, with

$$\begin{array}{lll} B_i = LA_{i+1} & \text{for} & i \equiv m-1 \pmod n, \\ B_i = A_{i-1} & \text{for} & i \equiv m \pmod n, \\ B_i = A_i & \text{for} & i \not\equiv m, m-1 \pmod n, \end{array}$$

is called a *left mutation* of a helix at A_m.

A collection $\{C_i,\ i \in \mathbf{Z}\}$, with

$$\begin{array}{lll} C_i = RA_{i-1} & \text{for} & i \equiv m+1 \pmod n, \\ C_i = A_{i+1} & \text{for} & i \equiv m \pmod n, \\ C_i = A_i & \text{for} & i \not\equiv m, m+1 \pmod n, \end{array}$$

is called a *right mutation* of a helix at A_m.

The basic fact about helices is the following.

THEOREM. *A right or left mutation of a helix is again a helix.*

All applications of helices are based on this theorem and the fact that any property of some given helix which is preserved by mutations will also hold for any mutation of that helix.

PROOF. (of the theorem) The symmetry of the definitions implies that it is enough to prove the theorem for a left mutation. In the proof we will consider collections $\{A_i \mid i \in \mathbf{Z}\}$ for which (hel) holds for some subset of the indices and not necessarily for them all.

LEMMA 1. *Let $\{A_i\}$ be a collection for which (hel) holds for $s \in S \subset \mathbf{Z}$ and let $\{B_i\}$ be obtained from $\{A_i\}$ by a left mutation at the point A_m, $m \in S$. Then $\{B_i\}$ is a helix for all $s \in \mathcal{L}_m S$, where*

$$\mathcal{L}_m S = \Big(S \setminus \{i \mid i \equiv m \pmod n)\}\Big) \cup \Big\{i \mid i \equiv m-1 \pmod n)\Big\}.$$

PROOF. As the choice of the position of a mutation is determined up to the period n, we can assume that $m-1 \leqslant s < m+n-1$. If $s = m-1$ then

$$\begin{aligned} &B_s = L^{(1)} A_m, \\ &(B_{s-1}, B_s) = (A_{m-2}, L^{(1)} A_m), \\ &\qquad \vdots \\ &(B_{s-n+1}, L^{(n-2)} B_s) = (A_{m-n-1}, A_{m-n}). \end{aligned}$$

The last equality holds because $L^{(n-2)}B_s = L^{(n-1)}A_m = A_{m-n}$. It is clear that $L^{(n-1)}B_s = L^{(1)}A_{m-n} = B_{s-1}$. We must now prove helicity for all $s \in S$ in the range $m+1 \leqslant s < m+n-1$. We see that $L^{(i)}A_s = L^{(i)}B_s$ for $i \leqslant s-m-1$. Put $A = A_{m-1}$, $B = A_m$, $C = L^{(s-m-1)}A_s$, and apply Axiom (2L). The conditions of the axiom are fulfilled because $s \in S$ and so we have helicity. Then the pairs (A, C) and $(B', L_A C)$ are equal to the pairs $(B_m, L^{(s-m-1)}B_s)$ and $(B_{m-1}, L^{(s-m)}B_s)$ respectively. We can see that they are left admissible and $L^{s-m+1}B_s = L^{s-m+1}A_s$. Now it is easy to see that such an equality is preserved and that all pairs arising from $\{B_i\}$ coincide with those for $\{A_i\}$. As a result $L^{(n-1)}B_s = L^{(n-1)}A_s = A_{s-n} = B_{s-n}$, and the lemma follows.

COROLLARY.

1) *Property (hel) is satisfied for all $s \in S$ if and only if (hel)′ is satisfied for all $s \in (S - \{n\})$.*
2) *If the collection $\{C_i\}$ is obtained from $\{A_i\}$ by right mutation at the point A_m, then $\{C_i\}$ satisfies (hel) for all $s \in \mathcal{R}_m S$, where*

$$\mathcal{R}_m S = \Big(S \setminus \{i \mid i \equiv m \,(\mathrm{mod}\, n)\}\Big) \cup \Big\{i \mid i \equiv m+1 \,(\mathrm{mod}\, n)\Big\}.$$

LEMMA 2. *If $\{A_i\}$ is a helix, then $\{B_i\}$ satisfies (hel) when $s \equiv m \,(\mathrm{mod}\, n)$.*

PROOF. We can assume that $s = m$. Note that since $m \in S$, the collection $\{B_i\}$ can be obtained from $\{A_i\}$ by the successive application of $n-2$ right mutations at the points with indices $m-n$, $m-n+1, \ldots, m-2$. Applying part (2) of the corollary we see that m is always left in the set of those indices for which (hel) is satisfied.

To complete the proof of the theorem we note that if $\{A_i\}$ is a helix for $S = \mathbb{Z}$ then, from Lemma 1 the collection $\{B_i\}$ is a helix for $s \not\equiv m \,(\mathrm{mod}\, n)$. But from Lemma 2, (hel) is also satisfied for these s. This proves the theorem.

2. Notation.

In the applications of the theory of helices there are a lot of computations with the functors Hom and Ext^i. In order to write these calculations down in a more compact form, we will use the notation

$${}^i\langle A \mid B \rangle = \mathrm{Ext}^i(A, B).$$

Generalizing the traditional notation in physics, we denote the functor $\mathrm{Ext}^i(A, -)$ by ${}^i\langle A \mid$, and the functor $\mathrm{Ext}^i(-, B)$ by ${}^i\mid B \rangle$. When applying such notation to the functor Hom we shall drop the "0" whenever this will not lead to confusion.

3. Exceptional Objects.

Let $\mathcal{F}$ be a vector bundle or a sheaf on an n-dimensional variety X over a field k. We call $\mathcal{F}$ *exceptional* if

$$^0\langle \mathcal{F} \mid \mathcal{F} \rangle = k, \quad \text{and} \quad ^i\langle \mathcal{F} \mid \mathcal{F} \rangle = 0 \quad \text{for} \quad 0 < i < n,$$

and the space $^n\langle \mathcal{F} \mid \mathcal{F} \rangle$ has the smallest possible dimension. More concretely, if X is a projective space, a quadric or any other variety with an ample anticanonical class then we must have $^n\langle \mathcal{F} \mid \mathcal{F} \rangle = 0$. If X is a K3 surface, an elliptic curve or any other variety with zero canonical class, then $^n\langle \mathcal{F} \mid \mathcal{F} \rangle = k$.

Rank one bundles are always exceptional, so we can express the fact that $\mathcal{F}$ is exceptional in the form

$$^i\langle \mathcal{F} \mid \mathcal{F} \rangle = {}^i\langle \mathcal{O}_X \mid \mathcal{O}_X \rangle \qquad \text{for all } i.$$

4. Mutations.

In studying exceptional bundles on varieties with ample anticanonical classes it seems to be convenient to use the notion of an exceptional collection [4].

By an *exceptional collection* we mean a collection of exceptional sheaves $F_1, \dots, F_s$ on X, such that for all $1 \leqslant l < m \leqslant s$,

$$\begin{aligned} &^i\langle F_m \mid F_l \rangle = 0 && \text{for} \quad i \geqslant 0, \\ &^i\langle F_l \mid F_m \rangle = 0 && \text{for all } i \geqslant 0, \text{ except for perhaps one value of } i \end{aligned}$$

We can now consider mutations of exceptional pairs. Let (A, B) be an exceptional pair. We say that this pair is left admissible with shift $L_A B$ if one of the following three possibilities occur:

(1) $^0\langle A \mid B \rangle \neq 0$ and the canonical map $^0\langle A \mid B \rangle \otimes A \to B$ is epimorphic. Then define $L_A B$ by
$$0 \to L_A B \to {}^0\langle A \mid B \rangle \otimes A \to B \to 0,$$

(2) $^0\langle A \mid B \rangle \neq 0$ and the canonical map $^0\langle A \mid B \rangle \otimes A \to B$ is monomorphic. Then define $L_A B$ by
$$0 \to {}^0\langle A \mid B \rangle \otimes A \to B \to L_A B \to 0.$$

(3) $^1\langle A \mid B \rangle \neq 0$. Then $L_A B$ is defined to be the universal extension
$$0 \to B \to L_A B \to {}^1\langle A \mid B \rangle \otimes A \to 0.$$

Case (1) is called *division*, case (2) is called *recoil* and case (3) is called *extension*.

Similarly, right mutations are defined from the following sequences:

$$0 \to A \to {}^0\langle A \mid B\rangle^* \otimes B \to R_B A \to 0 \qquad \text{division;}$$

$$0 \to R_B A \to A \to {}^0\langle A \mid B\rangle^* \otimes B \to 0 \qquad \text{recoil;}$$

$$0 \to {}^1\langle A \mid B\rangle^* \otimes B \to R_B A \to A \to 0 \qquad \text{extension.}$$

Where the last sequence is the universal extension.

In some cases, for example $X = \mathbf{P}^2$, one can show that all the mutations turn out to be divisions [4]. In the case $X = \mathbf{P}^n$, $n > 2$ in [4], this is made part of the definition and only such mutations are considered. In that paper some theorems are proved which verify the axioms of helix theory. In particular it is shown that the bundles $\{\mathcal{O}_{\mathbf{P}^n}(i)\}$ form a helix.

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2. Construction of Bundles on an Elliptic Curve

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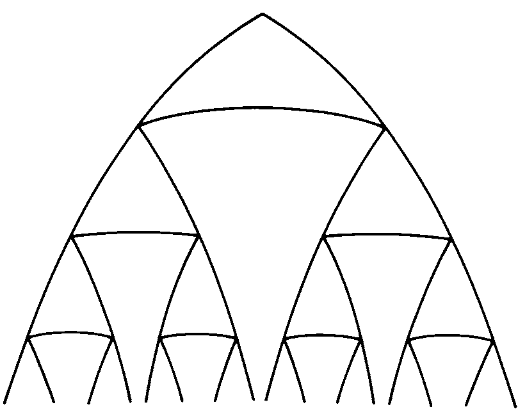
M.F. Atiyah [1] proves that the moduli space of indecomposable bundles with a fixed degree and rank over an elliptic curve is isomorphic to the curve itself. Atiyah also shows that an indecomposable bundle over an elliptic curve is simple if and only if the highest common factor of its degree and rank is equal to one.

In this paper Atiyah's results are proved using a recent technique whose power is frequently demonstrated in this collection of papers. We wish to stress that the proof given here of Atiyah's first two theorems (on the moduli space of indecomposable bundles) really only differs from Atiyah's proof in terminology. However, the proof of the theorem on simple bundles is fundamentally different from Atiyah's and the author hopes that it is more elegant.

Apart from proving Atiyah's theorems, this paper will classify the simple bundles whose determinant is equal to a multiple of a fixed point. The classification resembles that of exceptional bundles on the projective plane. In particular, we will construct the following graph:



In this graph each vertex represents an exceptional pair of bundles on an elliptic curve (see def. 2); each edge a mutation of an exceptional pair; and each smooth curve a simple bundle with non-negative degree. Moreover, every exceptional pair and every simple bundle, with non-negative degree and determinant equal to a multiple of a fixed point, is represented in this graph.

In this article we will confine ourselves to the following notation: X is a nonsingular elliptic curve over the field of complex numbers, $d(E)$ is the degree of a bundle E, equal to the degree of its determinant, $r(E)$ is the rank of the bundle E, $\mathcal{E}(r,d)$ is the set of indecomposable bundles over X of rank r and degree d.

1. Properties of indecomposable bundles over an elliptic curve.

In this section we formulate and prove two lemmas on the properties of indecomposable bundles over an elliptic curve, which we will need in the proof of Atiyah's theorem.

LEMMA 1. *Let X be a nonsingular elliptic curve, E an indecomposable vector bundle over X. If $h^0(E) \neq 0$ then there exists a filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E$$

with the following properties:

1) $E_i/E_{i-1} \cong L_i$, a line bundle,

2) $h^0(L_1) \neq 0$, $h^0(L_1^* \otimes L_i) \neq 0$, where $i \geqslant 1$.

PROOF. First we note that a filtration satisfying property (1) always exists. Indeed, for any bundle E there exists $n \in \mathbb{N}$ such that $E \otimes \mathcal{O}(n)$ is generated by its global sections, where $\mathcal{O}(n)$ as usual denotes the nth tensor power of a hyperplane section. Since a filtration of $E \otimes \mathcal{O}(n)$ induces a filtration of E, we can assume that E is globally generated. Consider a nonzero homomorphism $\phi : \mathcal{O} \to E$. Since its image is a torsion-free sheaf on a curve, it is locally free, i.e. it determines a rank one subbundle of E. We will denote this subbundle by $[\phi] = E_1$. Since a quotient sheaf of a globally generated sheaf is globally generated, we can again choose a rank one subbundle L_2 of the bundle E/E_1. We define a subbundle E_2 of E by the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_1 & \longrightarrow & E & \longrightarrow & E/E_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F_1 & = & F_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Continue by induction.

Next we observe that the set $\mathcal{D} = \{\deg[\phi] \mid \phi \in H^0(E),\ \phi \neq 0\}$ is bounded above. Indeed, consider some filtration of E satisfying (1) and a nonzero section $\phi \in H^0(E)$. There exists an integer $i \geqslant 1$ such that $[\phi] \subset E_i$, $[\phi] \not\subset E_{i-1}$ and ϕ defines a nonzero homomorphism $[\phi] \to L_i$, so $\deg[\phi] \leqslant \deg L_i$ and, since the set $\{\deg L_i \mid i = 1, \dots, r\}$ is bounded, the set $\mathcal{D}$ is bounded above.

We now proceed to construct a filtration satisfying (2). Choose E_1 to be a rank one subbundle of E of maximal degree. Clearly, $h^0(L_1) \neq 0$. Assume that we have already constructed the subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_i \subset E$$

satisfying (2) and such that L_j has maximal degree in E/E_{j-1}. Consider the exact sequence

$$0 \longrightarrow E_i \longrightarrow E \longrightarrow E_i' \longrightarrow 0.$$

As E is indecomposable, we have ${}^1\langle E_i' \mid E_i \rangle \neq 0$ and, from Serre duality, there exists a nonzero homomorphism $f : E_i \to E_i'$. There is an integer j, with $1 \leqslant j \leqslant i$, such that $f(E_{j-1}) = 0$ but $f(E_j) \neq 0$. Then f induces a nonzero homomorphism $\overline{f} : L_j \to E_i'$. From the induction hypothesis $h^0(L_j) > 0$. Consequently, the composite $(\overline{f} \circ \phi)$ of $\overline{f}$ with the nonzero section ϕ turns out to be a section of E_i'. Moreover, $h^0(L_j^* \otimes [\overline{f} \circ \phi]) \neq 0$. If $[\overline{f} \circ \phi]$ has maximal degree then we choose $L_{i+1} = [\overline{f} \circ \phi]$. Otherwise, we take L_{i+1} to be to be a subbundle of E_i' of maximal degree. Then $\deg L_{i+1} \geqslant \deg[\overline{f} \circ \phi] + 1$ and, as X is an elliptic curve, ${}^0\langle [\overline{f} \circ \phi] \mid L_{i+1} \rangle \neq 0$, i.e. we always have ${}^0\langle L_1 \mid L_{i+1} \rangle \neq 0$. To complete the proof, it remains to determine E_{i+1} from the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_i & \longrightarrow & E_{i+1} & \longrightarrow & L_{i+1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_i & \longrightarrow & E & \longrightarrow & E_i' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F_i & = & F_i & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We denote the filtration obtained in this lemma by $(L_1, L_2, \dots, L_r)$.

LEMMA 2. *Let* $E \in \mathcal{E}(r,d)$. *Then*

1) $\chi(E) = \deg E = d$.
2) *If* $\deg E < 0$, *then* $h^0(E) = 0$ *and* $h^1(E) = d$.
3) *If* $\deg E > 0$, *then* $h^0(E) = d$ *and* $h^1(E) = 0$.
4) *If* $d = 0$ *and* $h^0(E) > 0$, *then* $h^0(E) = 1$.

PROOF. Consider the filtration $(L_1, L_2, \ldots, L_r)$ of the bundle E. Using the additivity of the functions χ and deg, it is easy to check that

$$\sum_{i=1}^{r} \chi(L_i) = \chi(E) \quad \text{and} \quad \sum_{i=1}^{r} \deg L_i = \deg E.$$

Now, for line bundles, we have the Riemann-Roch formula $\chi(L_i) = \deg L_i$, so the first part of the lemma is clear.

Next, under the hypothesis of part (2), assume that $h^0(E) > 0$. Then we may require the filtration to satisfy $h^0(L_i) \neq 0$. But, since $\deg L_i \geqslant 0$, the degree of E is also non-negative, which contradicts the hypothesis. The third statement of the lemma follows easily from the second and Serre duality.

Now let the degree of E be equal to zero and $h^0(E) > 0$. Then the sum of the nonnegative degrees of the L_i (the elements of the filtration of the bundle E) is equal to zero and thus $\deg L_i = 0$ for all i. But all the L_i have sections, which is possible only when they are trivial.

We assume that E_i is the smallest element of the filtration satisfying the condition $h^0(E_i) \geqslant 2$, then $E_i \cong \mathcal{O} \oplus E_{i-1}$. Indeed, E_i is included in the exact sequence

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow \mathcal{O} \longrightarrow 0. \tag{1}$$

and the coboundary homomorphism $\delta : H^0(\mathcal{O}) \to H^1(E_{i-1})$ is zero. Therefore sequence (1) splits. The space $H^k(E_i)$ includes the distinguished line $H^k(E_{i-1})$ $(k = 0, 1)$. The direct summand $\mathcal{O}$ determines another line in $H^k(E_i)$ which is different from $H^k(E_{i-1})$. It is easy to see that any line $\ell \in H^k(E_i)$ different from $H^k(E_{i-1})$ determines a decomposition $E_i = E_{i-1} \oplus \mathcal{O}$ such that $H^k(\mathcal{O}) = \ell$.

For the next term in the filtration the space of global sections is not smaller than that of E_i. Let us assume that $h^0(E_{i+1}) = h^0(E_i)$. From the universal property there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_i & \longrightarrow & E'_{i+1} & \longrightarrow & H^1(E_i) \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow i & & \\
0 & \longrightarrow & E_i & \longrightarrow & E_{i+1} & \longrightarrow & \mathcal{O} & \longrightarrow & 0
\end{array} \tag{2}$$