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Cambridge University Press
978-0-521-38620-3 - Introduction to Uniform Spaces
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Excerpt
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1. Uniform structures

Let us start by recalling some notation and terminology from the theory of relations. Formally a relation on a given set X is just a subset R of the cartesian square $X \times X$. We write $\xi R \eta$ when $(\xi, \eta) \in R$ and say that ξ is R-related to η. Given ξ we denote by

$$R[\xi] = \{\eta : \xi R \eta\}$$

the set of R-relatives of ξ. For each subset H of X we write

$$R[H] = \bigcup_{\xi \in H} R[\xi] .$$

Note that if $\{H_j\}$ is a family of subsets of X then

$$R[\cup H_j] = \cup R[H_j];$$

in general, however, $R[\cap H_j]$ is a proper subset of $\cap R[H_j]$.

The identity relation on X is just the diagonal ΔX of $X \times X$. A relation R which contains ΔX is said to be reflexive. The reverse of a relation R is the relation R^{-1} given by

$$\xi R^{-1} \eta \text{ if and only if } \eta R \xi .$$

We describe R as symmetric if $R = R^{-1}$; for example the identity relation is symmetric.

The composition of relations R, S on the same set X is the relation $R \circ S$ on X given by $\xi(R \circ S)\eta$ if and only if $\xi S \zeta$ and $\zeta R \eta$ for some ζ. Note that $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. Also note that

$$(R \circ S)[H] = R[S[H]]$$

for each subset H of X. Composition of relations is associative (but generally not commutative) and so bracketing is unnecessary for repeated compositions such as

$$R^n = R \circ \ldots \circ R \quad (n \text{ factors}) .$$

Thus ξ is R^n-related to η when ξ is related to η by an R-chain of length n, i.e. a sequence $x_0, \ldots, x_n$ of points of X such that $x_0 = \xi$, $x_n = \eta$ and $x_i R x_{i+1}$ for $i = 0, \ldots, n-1$.

We describe R as transitive if $R \circ R \subset R$. If R is reflexive, symmetric and transitive then R is said to be an equivalence relation, and $R[\xi]$ is called the equivalence class of ξ. The set of equivalence classes is denoted by X/R and the surjection $\pi : X \to X/R$, given by $\pi(\xi) = R[\xi]$, is called the natural projection.

After these preliminaries we are ready for the definition of uniform structure on a given set X. This structure is a filter on the cartesian square $X \times X$ satisfying certain conditions. The members of the filter are not called relations (although the notations of relation theory are still used) but entourages.

Definition (1.1). A uniform structure on a given set X is a filter Ω on $X \times X$ such that

(i) $\Delta X \subset D$ for all $D \in \Omega$,

(ii) $D \in \Omega$ implies $D^{-1} \in \Omega$,

(iii) $D \in \Omega$ implies $E \circ E \subset D$ for some $E \in \Omega$.

Note that (iii) extends by iteration to the condition that for any $n \geq 1$ there exists an entourage E such that $E^n \subset D$. We shall use this extension without comment in what is to follow.

By a uniform space we mean a set X together with a uniform structure on X; usually X alone is sufficient notation. A refinement of a uniform structure Ω is a uniform structure Ω' such that each entourage of Ω is also an entourage of Ω'. In this situation we say that Ω' refines Ω, or that Ω coarsens Ω'. If the possibility that $\Omega = \Omega'$ is to be excluded we describe the refinement as strict.

Definition (1.2). The discrete uniform structure on a given set X is the structure in which every superset of the diagonal is an entourage.

In this situation we describe X as a discrete uniform space. Clearly the discrete uniform structure refines every other uniform structure. At the other extreme we have

Definition (1.3). The trivial uniform structure on a given set X is the structure in which the full set $X \times X$ is the sole entourage.

In this situation we describe X as a trivial uniform space. Clearly the trivial uniform structure is refined by every other uniform structure. Provided X has at least two distinct points the discrete uniform structure and the trivial uniform structure are different. When X has at least three distinct points there are other uniform structures as well.

If a filter is given by a base, so that the members of the filter are the supersets of the members of the base, we say that the base generates the filter. In the case of a uniform structure we say that the base generates the uniform structure and describe the members of the base as basic entourages. For example the symmetric entourages always form a base, a fact we shall be using frequently. The discrete uniform structure is generated by the diagonal.

Reversing our viewpoint, suppose that we have a base $\mathcal{B}$ for a filter Ω on $X \times X$. For Ω to be a uniform structure the base has to satisfy three conditions corresponding to those in (1.1), as follows.

(i) $\Delta X \subset D$ for all $D \in \mathcal{B}$,

(ii) $D \in \mathcal{B}$ implies $E \subset D^{-1}$ for some $E \in \mathcal{B}$,

(iii) $D \in \mathcal{B}$ implies $E \circ E \subset D$ for some $E \in \mathcal{B}$,

Apart from the modification to (ii), therefore, the conditions on $\mathcal{B}$ are the same as those on Ω itself.

For example, consider the real line $\mathbb{R}$. The three conditions are satisfied by the family of subsets

$$U_\epsilon = \{(\xi,\eta) : |\xi - \eta| < \epsilon\}$$

of $\mathbb{R} \times \mathbb{R}$, where ϵ runs through the positive reals. The uniform structure generated by this base is called the Euclidean uniform structure on $\mathbb{R}$. A similar uniform structure can be defined in the case of $\mathbb{R}^n$, or more generally as follows.

Recall that a pseudometric on a set X is a non-negative real-valued function $\rho : X \times X \to \mathbb{R}$ satisfying three conditions, as follows. First ρ vanishes on ΔX. Secondly, $\rho(\xi,\eta) = \rho(\eta,\xi)$, for all $\xi,\eta \in X$. Finally

$$\rho(\xi,\eta) \leq \rho(\xi,\zeta) + \rho(\zeta,\eta)$$

for all $\xi,\eta,\zeta \in X$. A pseudometric ρ determines a uniform structure on X by taking as base the family of subsets

$$U_\epsilon = \{(\xi,\eta) : \rho(\xi,\eta) < \epsilon\}$$

of $X \times X$, where ϵ runs through the positive reals.

For example the discrete pseudometric, which is constant and positive off the diagonal, determines the discrete uniform structure, while the trivial pseudometric, which is zero everywhere, determines the trivial uniform structure.

Of course different pseudometrics may determine the same uniform structure. For example the uniform structure determined by ρ is the same as the uniform structure determined by 2ρ.

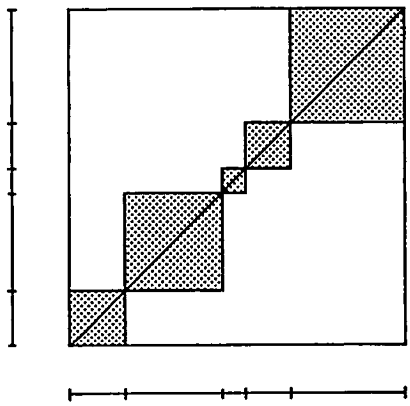
Observe that any uniform structure on X determined by a pseudometric ρ in this fashion admits a countable base, consisting of the family of entourages $\rho^{-1}[0,\epsilon)$, where ϵ runs through the positive rationals. In fact the converse is also true, as can be shown by the following construction, of which I omit the details since it is not required for anything we shall do later.

Suppose that X has a uniform structure with a countable base. Then we can construct inductively a base $\{D_n\}$ for the uniform structure $(n=0,1,\ldots)$ consisting of symmetric entourages D_n, with $D_0 = X \times X$ and $D_{n+1}^3 \subset D_n$ for all n. Next we define a real-valued function $\sigma : X \times X \to \mathbb{R}$ so that $\sigma(\xi,\eta) = 2^{-n}$ if there exists an integer n such that $(\xi,\eta) \in D_n \backslash D_{n+1}$ and $\sigma(\xi,\eta) = 0$ otherwise. Finally, the pseudometric $\rho : X \times X \to \mathbb{R}$ is defined by

$$\sigma(\xi,\eta) = \inf_{k>o}\{\Sigma_{i=0}^{k-1}\sigma(x_i,x_{i+1})\}$$

where the summation ranges over chains $x_1,\ldots,x_k$ of length k with $\xi = x_1$ and $\eta = x_k$.

If X is a uniform space the intersection R of the entourages constitutes an equivalence relation on X. For R is clearly reflexive and symmetric. To establish transitivity, let D be any entourage. Then $E \circ E \subset D$ for some entourage E. Since $R \subset E$ we have that $R \circ R \subset E \circ E$, hence $R \circ R \subset D$. This is true for every D and so $R \circ R \subset R$. Thus R is an equivalence relation, as asserted.



Basic entourage determined by partition.

When X is a finite set, in particular, the intersection R of the entourages is itself an entourage which therefore constitutes a base for the uniform structure. In fact the uniform structures correspond precisely to the equivalence relations, in the finite case, and these in turn correspond precisel to the partitions of X. Returning to the general case we make

Definition (1.4). A uniform structure on a given set X is separated if the diagonal Δ coincides with the intersection R of the entourages.

For example, if the uniform structure is determined by a pseudometric ρ, the structure is separated if and only if $\rho^{-1}(0) = \Delta$, i.e. if and only if ρ is a metric.

The class of separated uniform spaces is of great importance in practice. Two other classes of uniform space which are also important, are as follows.

Definition (1.5). The uniform space X is totally bounded if for each entourage D of X there exists a finite subset S of X such that $D[S] = X$.

Of course it is sufficient for the condition to be satisfied in the case of basic entourages when the uniform structure is generated by a base. For example, the trivial uniform structure is always totally bounded, while the discrete uniform structure is only totally bounded when X itself is a finite set.

To understand this important condition better let us introduce a further technical term. Given an entourage D of the uniform space X let us say that a subset M of X is small of order D, or simply D-small, if $M \times M \subset D$. Then the condition in (1.5) is that X can be covered by a finite number of D-small subsets, for each entourage D of X.

When the uniform structure is determined by a metric ρ on X then ρ must be bounded if X is totally bounded. For take D to be the basic entourage U_1 of X. The condition in (1.5) implies that no point of X is distance more than 1 from some point of a finite subset S_1 of X. Since S_1 is finite the diameter of S_1 is defined and then ρ is bounded by $2 + \operatorname{diam} S_1$.

Definition (1.6). The uniform space X is uniformly connected if for every entourage D of X, every pair of points of X is related by a D-chain.

In other words, for each pair of points ξ, η there exists an integer k such that $(\xi, \eta) \in D^k$. Of course it is sufficient if the condition is satisfied for basic entourages when the uniform structure is generated by a base. For example, the real line $\mathbb{R}$ is uniformly connected, with the Euclidean structure, more generally so is the real n-space $\mathbb{R}^n$. Also the rational line $\mathbb{Q}$ is uniformly connected, with the same structure. Since $\Delta^k = \Delta$ for all n we see that the discrete uniform structure is never uniformly connected, provided the space has more than one point. On the other hand the trivial uniform structure is always uniformly connected.

In the theory of uniform spaces the structure-preserving functions, in the inverse-image sense, are the uniformly continuous functions, defined as follows.

Definition (1.7). The function $\phi : X \to Y$, where X and Y are uniform spaces, is uniformly continuous if $(\phi \times \phi)^{-1}E$ is an entourage of X whenever E is an entourage of Y.

Thus ϕ is automatically uniformly continuous if the uniform structure of X is discrete or the uniform structure of Y is trivial.

Clearly the identity function on any uniform space is uniformly continuous. Also if $\phi : X \to Y$ and $\psi : Y \to Z$ are uniformly continuous, where X, Y and Z are uniform spaces, then the composition $\psi\phi : X \to Z$ is uniformly continuous. Thus uniform spaces form a category, with the uniformly continuous functions as morphisms. The equivalences of the category are called uniform equivalences.

Clearly it is sufficient for the condition in (1.7) to be satisfied for all basic entourages E, when the uniform structure of Y is generated by a base. When the uniform structures are determined by pseudometrics the condition can be expressed as follows.

Proposition (1.8). Let X and Y be uniform spaces where the uniform structures are determined by pseudometrics ρ and σ, respectively. Then a function $\phi : X \to Y$ is uniformly continuous if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\rho(\xi,\eta) < \delta$ implies $\sigma(\phi(\xi),\phi(\eta)) < \epsilon$, for all $\xi,\eta \in X$.

This, of course, is the condition which is familiar to all students of real analysis. When X and Y are metric spaces the distance-preserving functions are uniformly continuous. Isometries, which are distance-preserving bijections, are therefore uniform equivalences. The orthogonal transformations of $\mathbb{R}^n$ are examples of isometries.

A property of uniform spaces which is invariant under uniform equivalence is called a uniform invariant. The three properties we discussed earlier are all invariant in this sense. We now prove some results which help the behaviour of the last two to be better understood.

Proposition (1.9). Let $\phi : X \to Y$ be a uniformly continuous surjection, where X and Y are uniform spaces. If X is totally bounded then so is Y.

For let E be any entourage of Y. Then $D = (\phi\times\phi)^{-1}E$ is an entourage of X, since ϕ is uniformly continuous. Since X is totally bounded there exists a finite subset S of X such that $D[S] = X$. Then $\phi(D[S]) = \phi X = Y$. Since $\phi(D[S]) \subset E[\phi S]$, and ϕS is finite, this shows that Y is totally bounded, as asserted.

A similar result holds for uniform connectedness, as follows.

Proposition (1.10). Let $\phi : X \to Y$ be a uniformly continuous surjection, where X and Y are uniform spaces. If X is uniformly connected then so is Y.

This can easily be established directly or as a corollary of our next result, which provides a useful characterization.

Proposition (1.11). The uniform space X is uniformly connected if and only if for each discrete uniform space $\mathbb{D}$ every uniformly continuous function $\lambda : X \to \mathbb{D}$ is constant.

For suppose that X is uniformly connected. Given a uniformly continuous function $\lambda : X \to \mathbb{D}$, where $\mathbb{D}$ is discrete, consider the preimage $D = (\lambda\times\lambda)^{-1}\Delta$ of the diagonal Δ of $\mathbb{D}$. We have $D^n = D$, for all n, since $\Delta^n = \Delta$. Thus if $\xi,\eta \in X$ then $(\xi,\eta) \in D^n$, for some n, therefore $(\xi,\eta) \in D$ and so $\lambda(\xi) = \lambda(\eta)$. Consequently λ is constant.

Conversely suppose that X is not uniformly connected. Then there exists a symmetric entourage D of X and a pair $\xi,\eta \in X$ such that $(\xi,\eta) \notin D^k$ for all k. Taking $\mathbb{D} = \{-1,+1\}$ define $\lambda : X \to \mathbb{D}$ by $\lambda(x) = -1$ when $(\xi,x) \in D^i$ for some i and $\lambda(x) = +1$ otherwise. Then $D \subset (\lambda\times\lambda)^{-1}\Delta$ and so λ is uniformly continuous. Since λ is not constant this completes the proof.*

There is another way of forming a category where the objects are still uniform spaces but the morphisms are structure-preserving in the direct image sense, as follows.

Definition (1.12). The function $\phi : X \to Y$, where X and Y are uniform spaces, is uniformly open if for each entourage D of X there exists an entourage E of Y such that

$$E[\phi(x)] \subset \phi(D[x])$$

for all $x \in X$.

Of course it is sufficent if the condition is satisfied for basic entourages when the uniform structure of X is given by a base. For example ϕ is always uniformly open when Y is discrete. In the situation of (1.8) the condition

* There is an error in the corresponding result (9.34) of [10].