I

Isaac Newton's theory of a flattened earth (1687, 1713, 1726)

In the first edition of the *Principia* (1687), Isaac Newton reasoned that the earth cannot be a perfect sphere. Newton took the French astronomer Jean Richer's experiments with seconds pendulums in Cayenne, French Guyana, carried out in 1672–73, as the starting point that led him to conclude this. Richer found that a seconds pendulum had a shorter length in Cayenne than in Paris. To Newton this meant that the effective gravitational force per unit of mass, which I shall simply call the effective gravity hereafter, at the earth's surface had a smaller magnitude in the vicinity of the equator than in France. This Newton attributed in part to a decrease in the magnitude of the earth's centrifugal force of rotation with latitude. Newton assumed the earth initially to have been a fluid body, and he presupposed as well that in a fluid state the earth would be spherical, were it not for its rotation. He hypothesized that the centrifugal force of rotation caused the earth to flatten at its poles. As evidence to support his belief, Newton had John Flamsteed's and Gian-Domenico Cassini's telescopic observations of Jupiter's flattening at the poles.

I briefly sketch the version of Newton's theory of the earth's shape which appears in Book III of the third edition of the *Principia* (1726). By the 1730s, Parisian men of science were familiar with the third edition, and I shall be mainly concerned with scientific developments in Paris in the 1730s and 40s in the story that follows, although a considerable amount of background from earlier years must also be examined and discussed in connection with these developments, in order that they may be better understood. Since the values of the key parameters involved in the problem of the earth's shape do not change appreciably from one edition of the *Principia* to another, nothing is lost in restricting our attention to Newton's final presentation of his theory.[1]

Newton used the value of a degree of latitude between Amiens, France, and nearby Malvoisine, France, measured by the French astronomer Jean Picard in 1669–70, to determine the radius of an earth presumed to be homogeneous and spherical.[2] Knowing the earth's sidereal rate of diurnal rotation, Newton calculated the magnitude of the centrifugal force per unit of mass at the equator of the spherical earth. He also found the magnitude of the attraction, meaning the magnitude of the gravitational force per unit of mass, at the equator of the homogeneous spherical earth assumed to attract according to the universal inverse-square law in the following manner. Experiments made with seconds pendulums and with bodies falling freely in Paris furnished Newton with the magnitude of the component of effective gravity in Paris which is perpendicular to the earth's surface there. Knowing the latitude of Paris, Newton computed the magnitude of the component of centrifugal force per unit of mass in Paris perpendicular to the earth's surface there in terms of the magnitude of the centrifugal force per unit of mass at the equator of the spherical earth. Continuing to assume the earth to be spherical, Newton determined the magnitude of the attraction at the equator as follows: the magnitude of the attraction at the equator equals the magnitude of the attraction in Paris, which equals the magnitude of the component of effective gravity in Paris perpendicular to the earth's surface there plus the magnitude of the component of centrifugal force per unit of mass in Paris perpendicular to the earth's surface there. Newton then found the ratio of his calculated values of the magnitude of the centrifugal force per unit of mass and the magnitude of the attraction at the equator to be the fraction $\frac{1}{289}$.

(In 1690, Christiaan Huygens determined the same ratio to be this same fraction by hypothesizing that a central force of attraction, instead of Newton's universal inverse-square law of attraction, acts upon an earth assumed to be homogeneous and spherical, where Huygens supposed the center of force to be located at the center of the body that is attracted. In either case, the magnitude of the attraction along the surfaces of homogeneous *spherical* bodies is constant, while the variations of the magnitude of the effective gravity with latitude along the surfaces of Newton's and Huygens's spheres must be the same, since the two spheres have the same radii and rotate at the same rate. Hence Newton and Huygens arrived at the same fraction, $\frac{1}{289}$.)[3]

Newton then theorized that the earth was in reality a homogeneous body that attracts according to the universal inverse-square law and whose surface was an ellipsoid of revolution which is slightly flattened at its poles (see Figure 1.) He took the polar axis, which is the axis that the earth revolves around, to be the axis of symmetry of the ellipsoid of revolution. He postulated that the ratio of the magnitude of the centrifugal force per unit of mass at the equator to the magnitude of the attraction there remained $\frac{1}{289}$. Whatever the earth's actual shape, no one doubted that it differed only slightly from that of a sphere. Presumably then such a small deformation did not cause the ratio $\frac{1}{289}$ to change appreciably.[4]

To calculate the "ellipticity" $\varepsilon \equiv (E-P)/P$, a parameter that measures the degree of flattening or oblongness of a figure of revolution at its poles, where E is the equatorial radius of the figure and P is the polar radius of the figure, Newton assumed the earth to consist of homogeneous matter. He also supposed that two columns that join the center of the homogeneous body shaped like a flattened

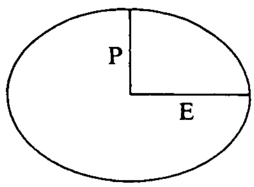

Figure 1. Meridian of a flattened ellipsoid of revolution.

ellipsoid of revolution to points at its surface and that lie along its polar and equatorial axes would balance. Expressed in terms of the calculations that Newton actually did, this assumption amounted to hypothesizing that the two columns weigh the same. Now, two columns that connect the center of a homogeneous solid body shaped like a flattened ellipsoid of revolution which revolves around its axis of symmetry (its polar axis) to points at its surface and which lie along the polar and equatorial axes of the body could certainly weigh the same. But what Newton really had in mind were two homogeneous fluid columns that he theorized would not displace each other. Two solid homogeneous columns could not displace each other, even if they did not weigh the same. Thus, as I mentioned, he treated the homogeneous rotating figure that he took to represent the earth to be a fluid figure, not a solid one. But in addition, in balancing two columns from the center to the surface of such a figure which lie along the polar and equatorial axes of the figure, Newton, in effect, not only treated the earth as a homogeneous rotating fluid figure, but he assumed as well that all of the columns from the center to the surface of the figure balance or weigh the same.

The polar axis of a flattened ellipsoid of revolution is its axis of symmetry. I note here for later purposes in Chapter 1, as well as in Chapters 4, 6, and 9, that in a homogeneous figure shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law and which revolves around its axis of symmetry, it is enough to consider columns from the center to the surface of such a figure which all lie in the same plane of a meridian at the surface of the figure without loss of generality. The reason is as follows. Suppose that two columns from the center to the surface of the figure do not lie in the same plane of a meridian at the surface of the figure. The point at the surface of the figure on either column must be located on some meridian at the surface of the figure, and the column in question of course lies in the plane of that meridian. But as a result of the symmetry of the figure of revolution, the fact that the axis of symmetry of the figure is also an axis of symmetry of the attraction produced by the figure, and the fact that the figure revolves around its axis of symmetry and not around some other line, so that the axis of symmetry of the figure is also an axis of symmetry of the centrifugal force per unit of mass, it follows that the other column can be revolved around the axis of symmetry of the figure into the plane of the meridian just defined without changing the length or the weight of this column.

The trouble with Newton's assumption that all of the columns from the center to the surface of a homogeneous figure shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law and which revolves around its axis of symmetry can balance or weigh the same is that he did *not* actually *verify* it. That is, he did *not* actually prove that all of the columns from the center to the surface of such a figure *can* balance or weigh the same. I shall return to this matter in Chapter 6.

Moreover, Newton in fact really assumed somewhat more. The polar axis of any figure of revolution is its axis of symmetry. Newton assumed that any homogeneous figure of revolution which revolves around its axis of symmetry and whose columns from center to surface all balance or weigh the same is a figure of relative equilibrium. As we shall see in Chapter 4, this need not be true in general. In Chapter 4 we shall discover that when homogeneous figures of revolution which revolve around their axes of symmetry are assumed to attract according to certain hypotheses of attraction, all of the columns from the centers to the surfaces of these figures can balance or weigh the same, yet these figures are not figures of relative equilibrium. Whether a homogeneous figure of revolution which revolves around its axis of symmetry and whose columns from center to surface all balance or weigh the same can be a figure of relative equilibrium or not depends upon the law according to which the figure attracts, as we shall observe in Chapter 4. A homogeneous figure shaped like an ellipsoid of revolution flattened at its poles which attracts according to the universal inverse-square law, which revolves around its axis of symmetry, and whose columns from center to surface all balance or weigh the same *does* turn out to be a possible figure of relative equilibrium, but we shall have to wait until Chapter 9 to see that this is true and to learn why it is true. I say a "possible" figure of relative equilibrium, because the ratio $\omega^2/2\pi\rho$ has a certain upper limit whose value is less than 1, where ω is the angular speed of rotation and ρ is the density of the figure. (In other words, for a given density ρ, the angular speed of rotation ω has an upper limit.) If the ratio $\omega^2/2\pi\rho$ exceeds this upper limit for a particular homogeneous figure shaped like an ellipsoid of revolution flattened at its poles which attracts in accordance with the universal inverse-square law and which revolves around its axis of symmetry (its polar axis), then such a figure cannot exist as a figure of equilibrium, even if the columns from the center of such a figure to its surface should all theoretically balance or weigh the same. None of the authors whose works I will discuss in this book took this upper limit into account. They did not realize that such an upper limit existed. The upper limit was first discovered only much later by Laplace.[5]

Now, Newton could also have assumed that the earth was originally a homogeneous fluid mass without having to assume that this mass existed in a state of equilibrium. However, by not assuming the homogeneous fluid mass to be in a state of equilibrium, Newton would have been left with no way of handling the problem of the earth's shape as he imagined it. But as we shall see in Chapters 4, 6, and 9, there are other ways to treat the problem of the earth's shape without having to assume the earth originally to have been a fluid mass, much less a figure of equilibrium. Specifically, we shall find that the earth can also be viewed instead as a solid, stratified figure that attracts according to the universal inverse-square law and that need not be a figure of equilibrium.

Finally, with regard to figures of equilibrium, among the homogeneous figures that attract according to the universal inverse-square law which Newton considered, he found ones that turn out to fulfill a particular condition that must be satisfied in order that homogeneous figures of equilibrium exist – namely, the principle of balanced columns, which means that all columns from the center of a homogeneous figure to its surface balance or weigh the same. (In Chapter 4 we shall see why the principle of balanced columns is a principle of equilibrium for homogeneous fluid figures of equilibrium, and in Chapter 9 we shall discover that the principle of balanced columns follows from another, more general principle of equilibrium for homogeneous fluid figures of equilibrium.) The part of the story told in these pages which has to do with figures of equilibrium mainly involves the discovery and use of various conditions that must be fulfilled in order that a fluid mass be a figure of equilibrium. Today we would call such conditions *necessary* conditions for figures of equilibrium to exist. The story does not deal at all with the problem of determining conditions that are *sufficient* for figures of equilibrium to exist.

The advantages of imagining the fluid figure that represented the earth to be shaped like an ellipsoid of revolution are evident. When the *Principia* first appeared, the tools of mathematical analysis had not yet been well developed. (Indeed, mathematical analysis only came into being at this time.) Newton could use Apollonian geometry as a guide in calculating attractions produced by homogeneous bodies shaped like flattened ellipsoids of revolution which attract according to the universal inverse-square law. He could then utilize these calculations to compute the weights of the two columns mentioned above, taking, of course, the effects of centrifugal force of rotation into account. In Proposition 91, Corollary 3 of Book I of the *Principia*, Newton stated that the magnitude of the attraction at points in the interior of a homogeneous body shaped like an ellipsoid of revolution which attracts in accordance with the universal inverse-square law varies directly as the distances of the points from the center of the body. If the body rotates as well and if the polar axis is its axis of symmetry and its axis of rotation, then the magnitude of the centrifugal force per unit of mass at points along an equatorial axis also varies directly as the distances of the points from the center of the body. (The magnitude of the centrifugal force per unit of mass equals zero at all points along the polar axis.) The two variations together make the magnitudes of the resultants of the effective gravity at points along the two axes vary directly as the distances of the points from the center of the body, too.[6]

Using this last result, Newton calculated the weights of two columns from the center of the body to its surface along its two axes. To be more specific, using Proposition 91, Corollary 2 of Book I of the *Principia*, Newton found the magnitude of the attraction at the equator of a homogeneous body shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law. Newton took $\frac{1}{289}$ to be the ratio of the magnitude of the centrifugal force per unit of mass to the magnitude of the attraction at the equator of the body when it revolves around its axis of symmetry (its polar axis), in which case $\frac{288}{289}$ is the ratio of the magnitude of the effective gravity to the magnitude of the attraction at the equator of the rotating body. This ratio multiplied by the magnitude of the attraction at the equator of the body equals the magnitude of the effective gravity at the equator of the body. Newton then utilized his result mentioned previously concerning the variation of the magnitude of the effective gravity along the body's equatorial axis to calculate the weight of a column that connects the equator of the body and its center. Similarly, after determining the magnitude of the attraction at a pole of a homogeneous body shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law, again by employing Proposition 91, Corollary 2 of Book I of the *Principia*, Newton used his result regarding the variation of the magnitude of the attraction in the body's interior to compute the weight of a column from a pole of the body to its center. If $\frac{1}{289}$ was taken to be the ratio of the magnitude of the centrifugal force per unit of mass to the magnitude of the attraction at the equator of the body when it revolves around its axis of symmetry, Newton discovered that the two columns weighed the same when the ellipticity ε of the body was equal to $\frac{1}{229}$.

Newton came to other conclusions as well. In so doing he made actual use of the assumption that *all* of the columns from the center to the surface of the figure that he had found balance or weigh the same. That is, Newton maintained that attraction and centrifugal force per unit of mass at points along any column from the center to the surface of a homogeneous figure shaped like a flattened ellipsoid of revolution which attracts in accordance with the universal inverse-square law and which revolves around its axis of symmetry determine how much that column weighs in exactly the same way that attraction and centrifugal force per unit of mass at points along two columns from the center of the figure to its surface which lie along its polar and equatorial axes determine how much those two columns weigh. Thus Newton concluded that the weight of any column from the center of a homogeneous figure shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law and which revolves around its axis of symmetry to a point at its surface was regulated in the same manner as the weights of two columns from the center of the figure to its surface which lie along its two axes. Applying this reasoning backward to an arbitrary column from the center of the figure to a point on its surface, and using the assumption that a short column weighs exactly the same as a long one, Newton deduced that the magnitude of the effective gravity at all points on the surface of his homogeneous figure shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law, which revolves around its axis of symmetry, and whose columns from center to surface were all assumed to balance or weigh the same varies inversely as the distances of the points from the center of the figure.[7] Consequently, he declared, although without furnishing the reader with any proof, that the increase in the magnitude of the effective gravity along the surface of the figure from its equator to one of its poles is nearly proportional to the square of the sine of the latitude and, moreover, that the increase in degrees of latitude with latitude is also nearly proportional to the square of the sine of the latitude.[8]

Newton used these findings to calculate theoretical values of the magnitude of the effective gravity and lengths of degrees of latitude at various latitudes along the earth's surface. He then compared his tables obtained in this way with the observations of the day made in Paris, Cayenne, Gorée (an island off the coast of Senegal, near Dakar), and the Caribbean. Newton noted that the magnitude of the component of effective gravity in Paris perpendicular to the earth's surface there exceeded the magnitudes of the components of the effective gravity in places at southern latitudes perpendicular to the earth's surface at those places, by which he meant that the lengths of seconds pendulums measured in Paris exceeded the lengths of seconds pendulums measured near the equator, by amounts greater than the ones that his theory required. Newton concluded that if these measurements could be trusted, then the earth was not homogeneous, but a little denser at its center than at its surface and a little flatter at its poles than was his homogeneous body shaped like a flattened ellipsoid of revolution.

(Newton, in fact, had greater faith in Richer's measurements than in other measurements made near the equator. He commented that if the observed difference between lengths of seconds pendulums in Paris and in Cayenne were reduced slightly to allow for lengthening of metallic cords in the Torrid Zone caused by the heat there, then the observed increase in the magnitude of the effective gravity with latitude modified in this way would agree closely with the increase in the magnitude of the effective gravity with latitude which his theory applied to the earth required, assuming that the earth is homogeneous.)

To affirm that Newton plainly demonstrated the various results and conclusions that his theory, outlined above, led him to would be to overstate the case. For Newton actually did no more than sketch his theory and its consequences himself. His theory differs considerably from what we would regard as a mathematical theory, and this fact probably explains in part why Derek Whiteside wrote so little in *The Mathematical Papers of Isaac Newton* about the theory. During the first fifty years that followed the publication of the *Principia*, Newton's theory of the earth's shape struck even the most reputable continental mathematicians of his time as incomprehensible.

I give some examples of the difficulties that the reader faced. As I mentioned earlier, Newton stated in Corollary 3 to Proposition 91 of Book I of the *Principia* that the magnitude of the attraction at points along any line segment from the center of a homogeneous body shaped like an ellipsoid of revolution which attracts according to the universal inverse-square law to a point on its surface varies directly as the distances of the points from the center of the body. But Newton's statement of this corollary is not precise, nor is his proof of what he did state complete.

In the first half of the proof, Newton arrived at the result that within the cavity of a homogeneous shell whose inner and outer surfaces are concentric geometrically similar ellipsoids of revolution having a common axis of symmetry and which attracts in accordance with the universal inverse-square law, the net attraction is zero. But his proof of this result has a gap: it depends upon the surprising equality of a certain pair of line segments, which Newton claimed to establish by invoking without proof a directly equivalent geometric statement (namely, that a certain other pair of line segments that lie along the same line have the same midpoint) that is not at all obvious.[9]

The second part of Newton's proof of Corollary 3 to Proposition 91 is based on Corollary 3 to Proposition 72 of Book I of the *Principia*. As it is stated by Newton, however, the meaning of this corollary is not immediately evident; and he chose not to present a proof of it – a proof which, had it been included, might have brought to full light the meaning of the corollary. Moreover, Proposition 72 is itself a difficult proposition, whose "meaning may remain uncertain even after several readings ..."; in its "excruciating terseness," the proof that Newton offers does little, unfortunately, to clarify the statement that it is supposed to prove.[10]

Finally, Newton never talked about the directions of attraction at points. But at each point there is the *total*, or *resultant of the*, attraction at the point, which has a certain well-determined direction. And then there are also *components* of the total attraction at the point in every *other* direction. To determine the weight of a column from the center of a figure to its surface one is interested in the magnitudes of the components of the attraction at points along the column which are directed toward the point at the surface. In fact, it can be shown, using methods patterned upon Newton's tersely stated arguments that make up his proof of Proposition 72 of Book I of the *Principia*, that the magnitudes of the *components* of attraction *in any fixed direction* at points along any line segment from the center of a homogeneous body shaped like an ellipsoid of revolution which attracts according to the universal inverse-square law to a point on its surface vary directly as the distances of the points from the center of the body. Furthermore, this result can be extended to cover the *total* attractions at points along any such line segment. That is, it can be demonstrated as well, again using Newton's proof of Proposition 72 of Book I of the *Principia* as a guide, that the *total* attractions at all points along a line segment from the center of a homogeneous figure shaped like an ellipsoid of revolution which attracts in accordance with the universal inverse-square law to a point on its surface *all have the same direction* and that their magnitudes vary directly as the distances of the points from the center of the figure. Indeed, it *follows immediately* from this last result that the magnitudes of the *components* of attraction *in any fixed direction* at points along any such line segment *must* vary directly as the distances of the points from the center of the figure.

Corollary 3 to Proposition 72 of Book I of the *Principia* can in fact be restated and proved in such a way that it gives, in conjunction with the result that Newton arrived at in the first half of his proof Corollary 3 to Proposition 91 of Book I of the *Principia*, all of these various results. It is possible that Newton realized this himself – that he could have stated this form of Corollary 3 to Proposition 72 of Book I of the *Principia* and simply did not. In order to do so, he would have had to introduce and explicate concepts like "magnitude of a force," "direction of a force," and "components of forces" distinguished from "total forces." Although he did not introduce such concepts, he may very well have understood them. But he did not bother to make such ideas clear for the benefit of his reader.[11]

And because Newton did not introduce and explain such concepts, one must greatly stretch the imagination to contend that Newton proved the different, specific results mentioned above, which require such concepts even to state them, in a way that readers of his time could understand these results and their proofs. He really did not even state the various results, much less establish them individually. As I observed, the different results and their proofs may have been clear to Newton. Newton possibly thought to himself a phrase that mathematicians today often use when teaching or giving talks: "It is obvious that . . . ." But what may have been obvious to Newton was almost never obvious to his contemporaries.

In practice, Newton used Corollary 3 to Proposition 91 of Book I of the *Principia* to infer that the magnitudes of the components of attraction at points along any line segment from the center of a homogeneous body shaped like an ellipsoid of revolution which attracts according to the universal inverse-square law to a point at its surface, *in the direction of that line segment*, vary directly as the distances of the points from the center of the body – although as I say, Newton in fact mentioned nothing in stating or proving this corollary which would lead one to conclude how the magnitudes of such *components* of attraction at points along the line segment actually vary. This particular result is what he used to reason about the way that the weights of columns from the center to points on the surface of a homogeneous figure shaped like an ellipsoid of revolution flattened at its poles which attracts in accordance with the universal inverse-square law and all of whose columns from center to surface are assumed to balance or weigh the same are regulated and to conclude that the magnitudes of the effective gravity at all points on the surface of such a figure vary inversely as the distances of the points from the center of the figure. As I indicated earlier, to arrive at this conclusion, Newton made direct use of the assumption that all of the columns from the center to the surface of the figure do balance or weigh the same.[12]

Moreover, Newton asserted that the ratio of the magnitude of the attraction at a pole to the magnitude of the attraction at the equator of a homogeneous body shaped like an ellipsoid of revolution which attracts according to the universal inverse-square law and whose polar radius is equal to $\frac{100}{101}$ of its equatorial radius (in other words, whose ellipticity is $\frac{1}{100}$) is $\frac{501}{500}$. As the basis for this conclusion, he cited Proposition 91, Corollary 2 of Book I of the *Principia,* mentioned above, which is a general, geometric theorem that can be used to determine the magnitudes of the attractions at points along the polar axes, in the exteriors, of homogeneous bodies that attract in accordance with the universal inverse-square law and whose surfaces are ellipsoids of revolution which are either flattened or elongated at their poles. (Here the attraction at a point on the polar axis of such a body in its exterior is the *total* attraction at the point produced by the body, which is directed along the polar axis because of symmetry.) But Newton stated the theorem without demonstrating it in detail.[13]

Then he maintained, again without going into details, that if this figure shaped like a flattened ellipsoid of revolution revolved around its axis of symmetry (its polar axis) and if columns from the center of the figure to its surface which lie along its polar and equatorial axes balanced or weighed the same, then the ratio of the magnitude of the centrifugal force per unit of mass to the magnitude of the attraction at the equator of this figure would be $\frac{4}{505}$.

Finally, he introduced, again without providing the reader with any demonstration, the equation

$$\frac{\delta}{1/100} = \frac{\phi}{4/505}, \tag{1.1}$$

from which he found $\frac{1}{229}$ to be the ellipticity δ of a homogeneous body shaped like a flattened ellipsoid of revolution which attracts according to the universal inverse-square law, which revolves around its axis of symmetry (its polar axis), and whose columns from center to surface which lie along the polar and equatorial axes of the body balance or weigh the same, when the ratio ϕ of the magnitude of the centrifugal force per unit of mass to the magnitude of the attraction at the equator of the body is the earth's value $\frac{1}{289}$. Newton did not explain where this equation came from or how he found it; he gave no account whatever of his procedure.

In fact, although Newton did not say so, this equation holds true *only* for a homogeneous figure shaped like a flattened ellipsoid of revolution which attracts in accordance with the universal inverse-square law, which revolves around its axis of symmetry (its polar axis), whose columns from center to surface all balance or weigh the same, and whose ellipticity is "infinitesimal," where an infinitesimal number ε is a nonzero number ε for which $\varepsilon^2 \ll \varepsilon$ is true. (To express the matter another way, equation (1.1) holds good only to terms of first order.) Newton must have known and understood this himself, considering the approximations that he had to have made in order to arrive at equation (1.1). Nevertheless, for some reason this did not stop him from applying equation (1.1) to Jupiter, as we shall see in Chapter 5, even though Newton knew from Flamsteed's and Gian-