London Mathematical Society Student Texts 18

Braids and Coverings: selected topics

VAGN LUNDSGAARD HANSEN
Mathematical Institute, Technical University of Denmark

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10, Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1989

Printed in Great Britain at the University Press, Cambridge

Library of Congress cataloguing in publication data: available

British Library cataloguing in publication data: available

ISBN 0 521 38479 6 Hardcover
ISBN 0 521 38757 4 Paperback

CONTENTS

Chapter I

BRAIDS AND CONFIGURATION SPACES

Braids are among the oldest inventions of mankind. They are used for practical purposes to make rope, and for decorations in weaving patterns and hairstyle etc.

As mathematical objects they were introduced by the German mathematician Emil Artin in a paper from 1925, although the idea was implicit in a paper of Hurwitz from 1891. Artin proposed to use braids to study knots and links. There are now applications of braids in several mathematical subjects: Topology (knots, links, fixed point theory, covering spaces), geometry (mapping class groups), singularity theory, dynamical systems, operator algebras, to mention a few. Such developments and many more were reported in a meeting on Artin's braid group held in Santa Cruz, July 13 to July 26, 1986, and has appeared in the conference proceedings (edited by J.S. Birman and A. Libgober) as volume 78 in the AMS Contemporary Mathematics series.

Chapter I contains the basic material on braid groups and the spaces related to them. The chapter opens with the definition of a geometric braid on n strings, first the original definition of Artin from 1925 as a system of n strings between two parallel planes in euclidean 3–space, and then the equivalent definition – suggested by Fox 1962 – as a loop in the space of configurations of a set of n points in the euclidean plane. The definition of Fox can be extended to define braids in any manifold and will prevail in the book. The fundamental system of fibrations of configuration spaces defined by Fadell and Neuwith 1962 plays an important role. We prove the presentation theorem of the Artin braid group on n strings in terms of generators and relations, and the representation theorem of this group as a subgroup of the group of automorphisms of the free group on n generators. In the final section of Chapter I we use braids in the 2–sphere to solve the Dirac string problem. It appears to be the first time this application of braid theory is included in a book.

1. Geometric braids.

Let $\mathbb{E}^3$ denote euclidean 3–space. We identify $\mathbb{E}^3$ with the 3–dimensional real number space $\mathbb{R}^3$ by choosing a coordinate system with coordinates (x,y,z), in which the Z–axis points vertically downwards. See Figure 1.

Consider two horizontal parallel planes in $\mathbb{E}^3$ with constant z–coordinates z_0 and z_1 respectively, where $z_0 < z_1$. We call the plane $z = z_0$ the upper plane and the plane $z = z_1$ the lower plane. Mark n different points $P_1,\ldots,P_n$ on a line in the upper plane and project them orthogonally onto the lower plane to the points $P'_1,\ldots,P'_n$.

Definition 1.1. A geometric braid on n strings (or an n–string braid, or just an n–braid) β is a system of n embedded arcs $\mathcal{A} = \{\mathcal{A}_1,\ldots,\mathcal{A}_n\}$ in $\mathbb{E}^3$, where the ith arc $\mathcal{A}_i$ connects the point P_i on the upper plane to the point $P'_{\tau(i)}$ on the lower plane for some permutation τ of $\{1,\ldots,n\}$, such that

(i) Each arc $\mathcal{A}_i$ intersects each intermediate parallel plane between the upper and the lower plane exactly once.

(ii) The arcs $\mathcal{A}_1,\ldots,\mathcal{A}_n$ intersect each intermediate parallel plane between the upper and the lower plane in exactly n different points.

The permutation τ is called the permutation of the braid. The arc $\mathcal{A}_i$ is called the ith string (or strand) in the braid.

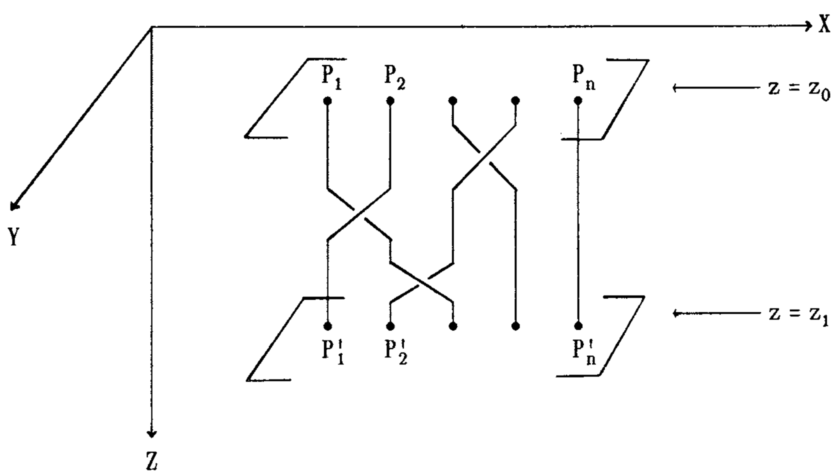


Figure 1

We think of an arc in $\mathbb{E}^3$ as the image of an embedding $\mathcal{A}_i$: $[0,1] \to \mathbb{E}^3$ of the unit interval $[0,1]$ into $\mathbb{E}^3$. We use the same notation for the arc and the corresponding embedding. As indicated, we think of a braid as hanging downwards.

To make braids into a useful concept we need to define equivalence of braids.

<u>Definition 1.2.</u> Two n–braids $\mathcal{A}^0 = \{\mathcal{A}^0_1,\ldots,\mathcal{A}^0_n\}$ and $\mathcal{A}^1 = \{\mathcal{A}^1_n,\ldots,\mathcal{A}^1_n\}$ with the same permutation τ, are called <u>equivalent</u> (or <u>homotopic</u>), if there is a homotopy through geometric braids with permutation τ from $\mathcal{A}^0$ to $\mathcal{A}^1$, in other words, if there exist n continuous maps

$$F_i: [0,1] \times [0,1] \to \mathbb{E}^3 \ , \ 1 \le i \le n \ ,$$

such that

$$\left.\begin{aligned} F_i(t,0) &= \mathcal{A}^0_i(t) \\ F_i(t,1) &= \mathcal{A}^1_i(t) \end{aligned}\right\} 0 \le t \le 1 \ , \ 1 \le i \le n$$

$$\left.\begin{aligned} F_i(0,s) &= P_i \\ F_i(1,s) &= P'_{\tau(i)} \end{aligned}\right\} 0 \le s \le 1 \ , \ 1 \le i \le n$$

and such that if we define $\mathcal{A}^s_i$: $[0,1] \to \mathbb{E}^3$ by $\mathcal{A}^s_i(t) = F_i(t,s)$, then $\mathcal{A}^s = \{\mathcal{A}^s_1,\ldots,\mathcal{A}^s_n\}$ is a geometric n–braid (with permutation τ) for each $0 \le s \le 1$.

<u>Remark 1.3.</u> There are several other possible notions of equivalence of geometric n–braids, e.g. ambient isotopy in $\mathbb{E}^3$ (through geometric n–braids) keeping the regions in $\mathbb{E}^3$ corresponding to $z \le z_0$ and $z \ge z_1$ pointwise fixed. As proved by Artin in 1947, [23], this notion leads to the same equivalence classes.

Throughout the remainder of the book, we shall not distinguish in notation between the equivalence class of a braid and the braid itself.

After a slight homotopy, we can (and will) assume that a braid β consists of polygonal arcs only, and that we get transversal crossings of the arcs if we project the braid orthogonally onto the plane in $\mathbb{E}^3$ containing the points $P_1,\ldots,P_n,P'_1,\ldots,P'_n$. By this projection we get a standard picture of the braid β as shown in Figure 2. Also note that (up to equivalence) we may assume that the crossings of strings occur on different levels, and that over and under crossings of strings must be indicated.

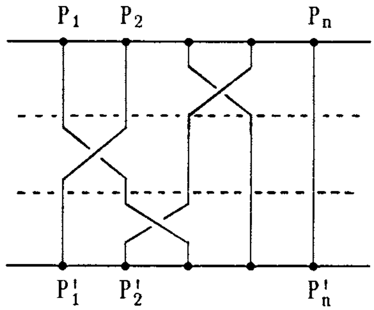


Figure 2

In Figure 2, we have indicated that a braid can be resolved into elementary braids, in which all strings except a neighbouring pair of strings go straight from the top to the bottom and the neighbouring pair just interchange.

For $1 \leq i \leq n-1$, we denote by σ_i that elementary geometric n–braid, in which the *i*th string just overcrosses the *(i+1)*th string once and all other strings go straight from the top to the bottom. See Figure 3.

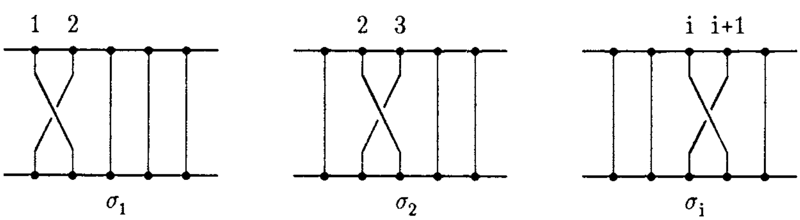


Figure 3

Let B(n) denote the set of all equivalence classes of geometric n–braids. It turns out that this set can be equipped with a natural group structure, which we shall now define.

Let β_1 and β_2 be geometric n–braids . Then we define the product (composition) of β_1 and β_2 , denoted $\beta_1 \cdot \beta_2$, as follows: First hang the braid β_2 under the braid β_1 by attaching the lower plane of β_1 to the upper plane of β_2 . Then remove the plane along which the braids β_1 and β_2 are attached to each other. Now squeeze the resulting system of arcs (strings) to lie between the planes $z = z_0$ and $z = z_1$, and we have the braid $\beta_1 \cdot \beta_2$. See Figure 4.

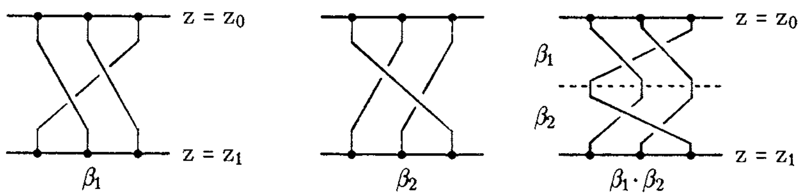
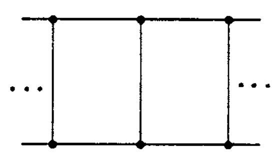


Figure 4

If we substitute homotopic braids β_1' and β_2' for β_1 and β_2 respectively, then it easy to prove that the product braids $\beta_1 \cdot \beta_2$ and $\beta_1' \cdot \beta_2'$ are homotopic. Thus the product is well defined on equivalence classes of n–braids and induces a product in B(n) .

The trivial n–braid ϵ is the n–braid in which all strings just go straight from the upper plane to the lower plane. The projection of ϵ is shown in Figure 5. It is easily seen that the equivalence class of ϵ is a neutral element for the product in B(n) .



Projection of the trivial n–braid ϵ

Figure 5

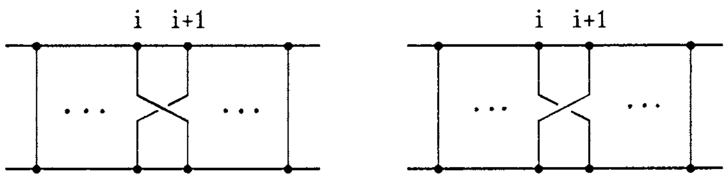
The inverse braid β^{-1} of the braid β is defined as the mirror image of β with respect to a horizontal plane between the upper plane and the lower plane. The projections of β and β^{-1} are shown in Figure 6.

β β^{-1}

Figure 6

It is easy to prove that the equivalence class of β^{-1} is well defined from that of β and that the product braids $\beta \cdot \beta^{-1}$ and $\beta^{-1} \cdot \beta$ are homotopic to the trivial braid. Therefore, the equivalence class of β^{-1} is the inverse element in B(n) to the equivalence class of β .

For the elementary n–braid σ_i , $1 \leq i \leq n-1$, the inverse braid σ_i^{-1} is obtained by changing (in the standard projection) the overcrossing of the i + 1 string by the i string to an undercrossing. See Figure 7.



σ_i σ_i^{-1}

Figure 7

It is now easy to prove that with the above product, neutral element and inverse elements, the set B(n) of equivalence classes of geometric n–braids is actually a group. This group is called the Artin braid group of braids on n strings. Since we shall prove in §3 that B(n) is the fundamental group of a suitable topological space, we shall not comment further on the existence of the group structure in B(n) at this point.

Remark 1.4. The 5–braid β with projection as in Figure 2 – or, more precisely, its equivalence class – can be written as the product $\beta = \sigma_3^{-1} \cdot \sigma_1^{-1} \cdot \sigma_2$. Notice that the elementary braid σ_i may occur in n–braids for all $n \geq i+1$. Hence it is important to know the number of strings in a braid β written as a product of elementary braids and their inverses.

As already indicated in Figure 2, it is intuitively clear that the equivalence class of any n–braid can be written as a product of the elementary n–braids σ_i , $1 \leq i \leq n-1$, and their inverses. In other words: The elementary n–braids $\sigma_1,\dots,\sigma_{n-1}$ generate the group B(n) .

We shall now look for relations among the elements in B(n) . First we notice that if $|i-j| \geq 2$ and $1 \leq i,j \leq n-1$, then – since the pair consisting of the i and i + 1 string does not interfere with the pair consisting of the j and j + 1 string – we get the following relation

$$(1) \quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \quad \text{for} \quad |i-j| \geq 2 \,,\, 1 \leq i \,,\, j \leq n-1 \;.$$

Figure 8 illustrates relation (1) .

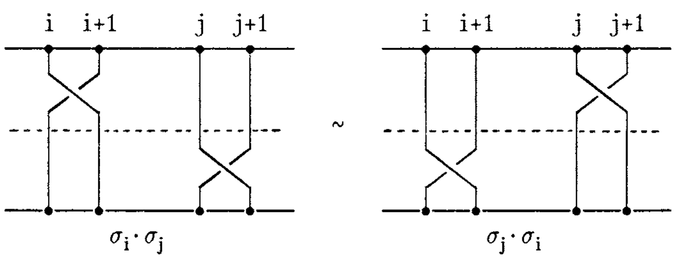


Figure 8

As illustrated in Figure 9, we also have the following relation in B(n)

$$(2) \quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2 \;.$$

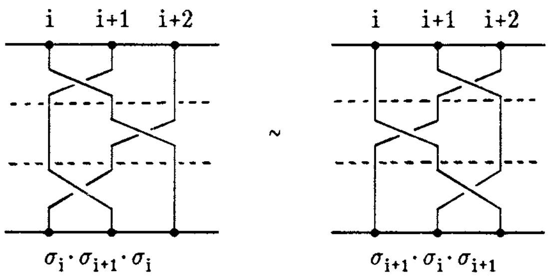


Figure 9

Already in Artin's first paper on braid groups [22] from 1925, a proof was given that the relations (1) and (2) generate all relations among the elements in $B(n)$. This is actually not quite trivial. We state the result below as Theorem 1.5, but postpone the proof until §4.

Theorem 1.5. The group $B(n)$ of geometric braids on n strings admits a presentation with

$$\text{generators: } \sigma_1, \sigma_2, \ldots, \sigma_{n-1}$$

and defining relations:

$$(1) \quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \quad \text{for} \quad |i-j| \geq 2 \ , \ 1 \leq i, j \leq n-1$$

$$(2) \quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \text{ for } 1 \leq i \leq n-2 \ .$$

2. Configuration spaces of ordered finite pointsets and their fibrations.

Let M denote a connected manifold of dimension ≥ 2. For an integer $n \geq 1$, consider the subset $F_n(M)$ in the n–fold product $M\times\ldots\times M$ of M with itself, consisting of all n–tuples of pairwise different points in M, i.e.

$$F_n(M) = \{(x_1,\ldots,x_n) \in M\times\ldots\times M \mid x_i \neq x_j \text{ for } i \neq j\} .$$

We can think of $F_n(M)$ as the space of possible configurations for a set of n ordered points in M. Accordingly, $F_n(M)$ is called the configuration space for a set of n (ordered) points in M. We equip $F_n(M)$ with the topology induced from the product topology on $M\times\ldots\times M$.

The configuration space $F_n(M)$ can be constructed from $M\times\ldots\times M$ by removing the finitely many submanifolds where any two coordinates are equal. Since the codimension of these submanifolds in $M\times\ldots\times M$ equals the dimension of M, and hence by assumption is ≥ 2, a simple transversality argument shows, therefore, that $F_n(M)$ is connected. Hence in particular, the homotopy groups of $F_n(M)$ are independent of base point.

Now let $m \geq 0$ be a nonnegative integer. By $Q_m = \{q_1,\ldots,q_m\}$ we denote a fixed, but arbitrary, subset of M consisting of m pairwise different points $q_i \in M$. For $m = 0$, take Q_0 to be the empty set. In order to study the configuration space $F_n(M)$, it is fruitful also to consider the configuration spaces

$$F_{m,n}(M) = F_n(M\backslash Q_m) .$$

If $Q'_m = \{q'_1,\ldots,q'_m\}$ is another subset of M containing m pairwise different points, then it is well known that there exists a homeomorphism on M taking Q_m to Q'_m. Hence the complements $M\backslash Q_m$ and $M\backslash Q'_m$ are homeomorphic open sets in M, and therefore the spaces $F_n(M\backslash Q_m)$ and $F_n(M\backslash Q'_m)$ are homeomorphic. Consequently, the topological type of the configuration space $F_{m,n}(M)$ is independent of the particular subset $Q_m \subset M$ chosen.

We note that

$$F_{o,n}(M) = F_n(M) \quad \text{and} \quad F_{m,1}(M) = M\backslash Q_m .$$

The following fundamental theorem was proved by Fadell and Neuwirth [30] in 1962.

Theorem 2.1. Suppose $n \geq 2$ and let $1 \leq r < n$. Then the map

$$\pi : F_{m,n}(M) \to F_{m,r}(M) \ ,$$

defined by

$$\pi(x_1,...,x_n) = (x_1,...,x_r) \ ,$$

is a locally trivial fibration with fibre $F_{m+r,n-r}(M)$.

Proof Consider a point $(x_1^0,...,x_r^0) \in F_{m,r}(M)$.

First we shall describe the inverse image under π of this point, $\pi^{-1}(x_1^0,...,x_r^0)$, which eventually will be the typical fibre of the fibration π . Clearly,

$$\pi^{-1}(x_1^0,...,x_r^0) = \{(x_1^0,...,x_r^0 \, , y_1,...,y_{n-r}) \mid \text{All coordinates different and in } M\backslash Q_m \} \ .$$

If we put $Q_{m+r} = Q_m \cup \{x_1^0,...,x_r^0\}$, then

$$F_{m+r,n-r}(M) = \left\{ (y_1,...,y_{n-r}) \;\middle|\; \begin{array}{ll} y_i \neq y_j & \text{for } i \neq j \\ y_i \in M\backslash Q_{m+r} & \end{array} \right\} \ ,$$

and there is an obvious homeomorphism

$$h : F_{m+r,n-r}(M) \to \pi^{-1}(x_1^0,...,x_r^0)$$

defined by

$$h(y_1,...,y_{n-r}) = (x_1^0,...,x_r^0 \, , y_1,...,y_{n-r}) \ .$$

The proof of local triviality of π will be carried out only for $r = 1$, mainly for notational convenience. For the rest of the proof, we shall therefore only consider the map

$$\pi : F_{m,n}(M) \to F_{m,1}(M) \ ,$$

defined by $\pi(x_1,...,x_n) = x_1$.

We shall prove that π is trivial over a neighbourhood U of an arbitrary point $x_0 \in F_{m,1}(M) = M\backslash Q_m$. As above we let $Q_{m+1} = Q_m \cup \{x_0\}$.

Let U denote a neighbourhood of x_0 in the open set $M \backslash Q_m$ in M, which is homeomorphic to an open ball, and let $\bar{U}$ denote the closure of U. Define a (continuous) map $\Theta : U \times \bar{U} \to \bar{U}$ with the following properties. Setting $\Theta_x(y) = \Theta(x,y)$ we require:

(i) $\Theta_x : \bar{U} \to \bar{U}$ is a homeomorphism, which fixes the boundary $\partial\bar{U}$ of $\bar{U}$ pointwise.

(ii) $\Theta_x(x) = x_0$.

In Sublemma 2.2 below, we shall prove that such a map Θ does exist.

By property (i), Θ can be extended to a (continuous) map $\Theta : U \times M \to M$ by setting $\Theta(x,y) = y$ for $y \notin U$.

The homeomorphism $\Theta_x : M \to M$ takes pairwise different $(n-1)$ – tuples of points in M avoiding $x \in U$ into pairwise different $(n-1)$ – tuples of points in M avoiding $x_0 \in U$. In other words, Θ_x maps the "fibre" of π over x homeomorphically onto the "fibre" of π over x_0. Then it follows easily that the required local product representation of π,

$$U \times F_{m+1,n-1}(M) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \pi^{-1}(U) \quad ,$$

is given by

$$\Phi(x, y_1,\dots,y_{n-1}) = (x, \Theta_x^{-1}(y_1),\dots,\Theta_x^{-1}(y_{n-1}))$$

$$\Phi^{-1}(x, y_1,\dots,y_{n-1}) = (x, \Theta_x(y_1),\dots,\Theta_x(y_{n-1})) \quad .$$

To finish the proof of Theorem 2.1, it only remains to prove

Sublemma 2.2. There exists a continuous map $\Theta : U \times \bar{U} \to \bar{U}$ with the properties (i) and (ii) as above.

Proof Let $\mathbb{R}^n$ denote the n–dimensional real number space with euclidean norm $\|\cdot\|$. To prove Sublemma 2.2, it is sufficient to consider the unit ball U in $\mathbb{R}^n$ and $x_0 = 0$. Then $\bar{U} = \{y \in \mathbb{R}^n \mid \|y\| \leq 1\}$.

For $x \in U$, i.e. $\|x\| < 1$, choose a function

$$\lambda_x : \bar{U} \to \mathbb{R}_0^+ \ ,$$

where $\mathbb{R}_0^+$ denotes the nonnegative real numbers, such that

$$\lambda_x(y) = \begin{cases} 1 & \text{for } 0 \le \|y\| \le \|x\| + \frac{1}{3}(1 - \|x\|) \\ 0 & \text{for } \|x\| + \frac{2}{3}(1 - \|x\|) \le \|y\| \le 1 \end{cases},$$

and such that $\lambda : U \times \bar{U} \to \mathbb{R}_0^+$ defined by $\lambda(x,y) = \lambda_x(y)$ is continuously differentiable.

For $x \in U$ define now a vector field v_x on $\bar{U}$ by

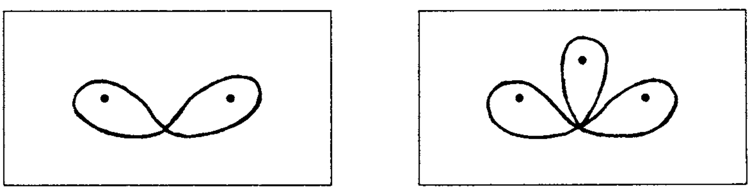
$$v_x(y) = \lambda_x(y) \cdot (0 - x) ,$$

and let $\Theta_x^t(y)$, $t \in \mathbb{R}$, be the corresponding flow. Then the homeomorphism $\Theta_x(y) = \Theta_x^1(y)$ is as desired. □

We shall now consider the special case, where we take the manifold M to be the euclidean plane $\mathbb{E}^2$.

First note that the space $\mathbb{E}^2 \backslash Q_m$ for $m \geq 1$ has the homotopy type of a wedge (bouquet) of m copies of the 1-sphere S^1 . See Figure 10.

$\mathbb{E}^2$
$m = 2$



$\mathbb{E}^2$
$m = 3$

Figure 10

In particular we note that all homotopy groups of $\mathbb{E}^2 \backslash Q_m$ in dimensions $i \geq 2$ vanish. From Theorem 2.1, we therefore deduce the following

Corollary 2.3 (Fadell and Neuwirth). For the configuration space $F_n(\mathbb{E}^2)$ of n ordered points in the plane $\mathbb{E}^2$, we have

$$\pi_i(F_n(\mathbb{E}^2)) = 0 \text{ for } i \geq 2 .$$

More generally: For all $n \geq 1$ and $m \geq 0$ we have

$$\pi_i(F_{m,n}(\mathbb{E}^2)) = 0 \text{ for } i \geq 2 .$$

<u>Proof</u> For notational simplicity, put $F_{m,n} = F_{m,n}(\mathbb{E}^2)$, and consider the following diagram

$$\begin{array}{ccccccccccc} F_{n-1,1} & \to & F_{n-2,2} & \to & \cdots & \to & F_{2,n-2} & \to & F_{1,n-1} & \to & F_{0,n} = F_n(\mathbb{E}^2) \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ & & F_{n-2,1} & & & & F_{2,1} & & F_{1,1} & & F_{0,1} \end{array} ,$$

in which the vertical maps are the fibrations of Theorem 2.1, and the horizontal maps are the inclusions of the fibres in the corresponding total spaces of the fibrations.

For the spaces $F_{m,1}(\mathbb{E}^2) = \mathbb{E}^2 \backslash Q_m$, all homotopy groups in dimensions $i \geq 2$ vanish. Working through the homotopy sequences for the fibrations, from the left to the right in the diagram, we conclude in the final stage that

$$\pi_i(F_n(\mathbb{E}^2)) = \pi_i(F_{0,n}) = 0 \text{ for } i \geq 2 .$$

More generally, as an intermediate step towards proving that $\pi_i(F_{0,n+m}) = 0$, we prove that

$$\pi_i(F_{m,n}(\mathbb{E}^2)) = \pi_i(F_{m,n}) = 0 \text{ for } i \geq 2 . \qquad \square$$

3. The braid group as fundamental group of a configuration space.

As in §2, let M denote a connected manifold of dimension ≥ 2.

Let Σ_n denote the symmetric group on n elements, i.e. the group of all permutations of the set $\{1,...,n\}$.

There is a natural right action of Σ_n on the configuration space $F_n(M)$,

$$\mu : F_n(M) \times \Sigma_n \to F_n(M) ,$$

defined by permutation of coordinates, i.e.

$$\mu((x_1,...,x_n),\sigma) = (x_1,...,x_n) \cdot \sigma = (x_{\sigma(1)},...,x_{\sigma(n)}) .$$

It is easy to prove that μ is indeed a right action. Since all coordinates for points in $F_n(M)$ are pairwise different, it is, moreover, a free action.

Denote the orbit space for the free action μ by $C_n(M)$, or in standard notation,

$$C_n(M) = {F_n(M)}/{\Sigma_n} .$$

The orbits of elements in $F_n(M)$ under the action μ consist of n–tuples $(x_1,..,x_n)$ of pairwise different points $x_i \in M$, where two n–tuples are on the same orbit if they differ only by a permutation. Hence, one can think of $C_n(M)$ as the space of all possible configurations for an unordered set of n pairwise different points in M. Accordingly, $C_n(M)$ is called the <u>configuration space</u> for a set of n (unordered) points in M.

Projection of $F_n(M)$ onto the orbit space $C_n(M)$ defines a principal Σ_n–bundle,

$$p_n : F_n(M) \to C_n(M) .$$

In particular, p_n is an n!–fold covering map.

Following an idea of Fox [31], configuration spaces can be used to define a notion of braids in M , which specializes to the geometric braids defined by Artin in the case $M = \mathbb{E}^2$. This will now be explored.

Choose a base point $\bar{c}_0$ in $F_n(M)$, and take $c_0 = p_n(\bar{c}_0)$ as base point in $C_n(M)$. Consider an arbitrary element $\beta \in \pi_1(C_n(M),c_0)$ in the fundamental group of $C_n(M)$, represented by the closed loop $f: [0,1] \to C_n(M)$ with $f(0) = f(1) = c_0$. There is a unique lifting of f in the covering map $p_n : F_n(M) \to C_n(M)$ to a path

$\bar{f}: [0,1] \to F_n(M)$ in $F_n(M)$ with $\bar{f}(0) = \bar{c}_0$. One can think of $\bar{f} = (\bar{f}_1,...,\bar{f}_n)$ as n paths $\bar{f}_i: [0,1] \to M$, $1 \leq i \leq n$, in M, for which $\bar{f}_i(t) \neq \bar{f}_j(t)$ for all $i \neq j$ and all $t \in [0,1]$. Note that the ordered set of points $(\bar{f}_1(1),...,\bar{f}_n(1))$ in M is just a permutation τ of the ordered set of points $(\bar{f}_1(0),...,\bar{f}_n(0))$ in M, since $p_n(\bar{f}(1)) = p_n(\bar{f}(0)) = c_0$.

Define embeddings $\mathcal{A}_i: [0,1] \to M \times [0,1]$, $1 \leq i \leq n$, by $\mathcal{A}_i(t) = (\bar{f}_i(t), t)$. The system $\mathcal{A} = \{\mathcal{A}_1,..,\mathcal{A}_n\}$ is then a system of n strings in the product space $M \times [0,1]$ satisfying the same formal requirements as laid down in Definition 1.1 for a geometric braid. In particular, the system of strings $\mathcal{A}$ in $M \times [0,1]$ connects the set of points $P_1,...,P_n$ in $M = M \times \{0\}$, which is the set of coordinates for $\bar{c}_0 \in F_n(M)$, to the corresponding set of points $P'_1,...,P'_n$ in $M \times \{1\}$, according to the permutation τ that turns $\bar{f}(0)$ into $\bar{f}(1)$.

The homotopy class of $\mathcal{A}$ defined in analogy with Definition 1.2 is well defined since it only depends on β and not on the particular representative $f: [0,1] \to C_n(M)$ originally chosen.

Following Fox, we call $\mathcal{A}$ – or , for the sake of simplicity, the original homotopy class $\beta \in \pi_1(C_n(M),c_0)$ itself – a braid with permutation τ in M on n strings, and the group $\pi_1(C_n(M),c_0)$ the group of braids on n strings in M.

The group $\pi_1(F_n(M),\bar{c}_0)$ is called the Fox group of coloured (or pure) braids on n strings in M . Coloured braids are exactly the braids with trivial permutation of the strings.

Now we specialize and take the manifold M to be the euclidean plane $\mathbb{E}^2$. We shall consider $\mathbb{E}^2$ as embedded in euclidean space $\mathbb{E}^3$, such that after identification of $\mathbb{E}^3$ with $\mathbb{R}^3$, the plane $\mathbb{E}^2$ corresponds to $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$.

Choose points

$$P_1 = (1,0,0)\ ,\ P_2 = (2,0,0),....,P_n = (n,0,0) \quad \text{in the plane } z = 0$$

and

$$P'_1 = (1,0,1)\ ,\ P'_2 = (2,0,1),....,P'_n = (n,0,1) \quad \text{in the plane } z = 1\ .$$

Let

$$p_n: F_n(\mathbb{E}^2) \to C_n(\mathbb{E}^2)$$

be the canonical pricipal Σ_n – bundle of configuration spaces, and take $\bar{c}_0 = (P_1,...,P_n) \in F_n(\mathbb{E}^2)$ as base point in $F_n(\mathbb{E}^2)$ and $c_0 = p_n(\bar{c}_0) \in C_n(\mathbb{E}^2)$ as the corresponding base point in $C_n(\mathbb{E}^2)$.

Then the braid $\mathscr{A}$ in $\mathbb{E}^2$ in the sense of Fox corresponding to a homotopy class $\beta \in \pi_1(C_n(\mathbb{E}^2),c_0)$ as defined above is clearly a geometric braid on n strings in the sense of Artin as defined in Definition 1.1. Conversely, it is also clear how to define a Fox braid from an Artin braid.

It is easy to prove that the product of equivalence classes of geometric n–braids defined in §1 corresponds to the standard product of homotopy classes of loops in $C_n(\mathbb{E}^2)$. Also formation of inverses in the two algebraic structures correspond to each other. Hence we get

Theorem 3.1. The Artin braid group $B(n)$ can be canonically identified with the fundamental group $\pi_1(C_n(\mathbb{E}^2),c_0)$.

In particular, we now have a full proof that $B(n)$ is actually a group. In the following, we shall freely identify $B(n)$ and $\pi_1(C_n(\mathbb{E}^2),c_0)$ when convenient.

The homotopy sequence for the principal Σ_n – bundle (n!–fold covering map) p_n: $F_n(\mathbb{E}^2) \to C_n(\mathbb{E}^2)$ reduces by Corollary 2.3 to the following short exact sequence:

$$1 \to \pi_1(F_n(\mathbb{E}^2),\bar{c}_0) \xrightarrow{\rho_n} \pi_1(C_n(\mathbb{E}^2),c_0) \xrightarrow{\tau_n} \Sigma_n \to 1 \ .$$

If we use the identification

$$B(n) = \pi_1(C_n(\mathbb{E}^2),c_0) \ ,$$

and put
$$H(n) = \pi_1(F_n(\mathbb{E}^2),\bar{c}_0) \ ,$$

then the elements in $H(n)$ can be identified with the (equivalence classes of) geometric braids on n strings with trivial permutation of the strings. As mentioned earlier, these braids are called coloured, or pure, braids.

The boundary homomorphism τ_n in the above short exact sequence is defined by lifting in the covering map p_n , and hence $\tau_n(\beta)$ is the permutation of the braid β for every braid $\beta \in B(n)$. The homomorphism ρ_n is induced by the map p_n.

Introducing these identifications in the above short exact sequence, we get the short exact sequence