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Part A.

Classical relativity and gravitation theory

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Colliding waves in general relativity

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1. Introduction

The gravitational interaction between waves is a phenomenon in which the richness and the originality of the theory of general relativity are explicitly manifested. It became apparent in 1970-71 when Khan, Penrose[1] and Szekeres[2] found the first exact solutions describing the collision of pure gravitational waves: it was shown that when two plane gravitational waves with collinear polarization, and with a step or an impulsive profile collide, their subsequent interaction culminates in the creation of a curvature singularity, an event unpredicted by any linearized version of the theory of gravity. As we shall see, this is only a particular result, although probably the most remarkable, of the interaction of gravitational waves. Similar behaviors are also manifested when waves of a different nature collide. This is due to the fact that any kind of energy generates a gravitational field. As a consequence, when two arbitrary waves collide, a gravitational interaction will accompany, as a side effect, the interaction which is peculiar to the particular fields considered. These gravitational effects, though negligible to some extent, are nevertheless relevant from a theoretical point of view. In this lecture we shall investigate the main features of the scattering of plane waves in terms of exact solutions of Einstein's equations. Therefore, let us start by explaining what gravitational plane waves are and how to find exact solutions of Einstein's equations describing their interaction. The methods I shall describe can be generalized when one is dealing with other kinds of null fields, as for example, electromagnetic or massless scalar fields.

2. Colliding wave solutions

A gravitational plane wave is a region of spacetime confined between two parallel planes, in which the curvature is different from zero and which propagates through the spacetime, in the direction normal to the planes, at the speed of light. This is usually referred to as a "sandwich wave." When only one wave is present, the spacetime is flat before and after the passage of the wave and curved inside the sandwich. Since we are considering vacuum solutions, the Ricci tensor is zero everywhere. More rigorously, a plane wave is a non-flat solution of Einstein's equations in vacuum, which admits a five parameter group of motions,[3] namely the same symmetries of an electromagnetic wave.

When the thickness of the sandwich tends to zero, the Riemann tensor remains finite on the hypersurface perpendicular to the direction of propagation and the Ricci tensor remains zero everywhere, the sandwich wave becomes an impulsive wave, and the corresponding Riemann tensor becomes proportional to a δ-function:

$$R^{\alpha}_{\beta\delta\gamma} \sim \delta(x-t)\,,$$

where x is the direction of propagation. This is the most simple model of a gravitational plane wave. It is apparent that these waves are idealized models. Firstly they are plane, thus they represent to some approximation, the field far from radiating sources. Secondly, they have an infinite wavefront. This assumption certainly imposes severe restrictions on the global behavior of the solutions which describe their collision. However, these solutions provide interesting information on the role played by the nonlinearity of Einstein's equations in these scattering processes, and they should act as a guide for the investigation of more realistic situations.

As previously mentioned, the first exact solutions describing the interaction of gravitational waves date back to 1970 and 1971. In 1972 Szekeres[4] showed how to state the problem of colliding waves as an initial data problem, in the case of collinear polarization. In 1977 Nutku and Halil[5] generalized the Khan-Penrose solution to the case when the impulsive waves have non-collinear polarization. In 1984 much attention has again been focused on these problems, due to an alternative approach suggested by Chandrasekhar and the author.[6] We showed that the mathematical theory of colliding waves can be constructed in a way similar to the mathematical theory of black holes, due to certain reciprocal relations existing between stationary axisymmetric spacetimes and spacetimes with two spacelike Killing vectors. The application of this theory, whose main features I shall briefly outline, allows us to find exact solutions describing the region where two plane waves interact, and it has been applied successfully during the past five years in obtaining a variety of new solutions.

Let us assume that two plane waves travel along the same direction x, one against the other. Due to the symmetry of the problem, the metric is expected to depend on $t \pm x$ only, and to be independent of y and z, which are assumed to be the coordinates on the wavefront. In other words, we require that the solution possess two spacelike Killing vectors, $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, which span the wavefront. With these assumptions, a suitable choice of the gauge allows the metric to be cast in the following form:

$$ds^2 = f\left[\frac{dt^2}{1-t^2} - \frac{dx^2}{1-x^2}\right] - \sqrt{(1-x^2)(1-t^2)}\left(\left[\frac{1-|E|^2}{|1-E|^2}\right]dy^2 + \frac{|1-E|^2}{1-|E|^2}\left[dz + i\frac{(E-E^*)}{|1-E|^2}dy\right]^2\right), \tag{1}$$

where $f = f(t \pm x)$ is real and $E = E(t \pm x)$ is a complex function. Thus, by the exclusive use of the symmetries, the number of unknown components of the metric tensor is reduced to 3. E and f must be found by solving Einstein's field equations in vacuum. Since we are considering waves travelling along the same direction, namely the "head-on" collision between plane waves, one might question whether or not this assumption is too restrictive. The answer is no, since if the two waves propagate along arbitrary directions, it is always possible to make a transformation to a frame of reference in which they appear to approach each other from exact opposite spatial directions. Therefore, the case of the head-on collision we are considering is not restrictive.

It is possible to demonstrate that the entire set of Einstein's equations splits into two blocks. E satisfies a **non-linear** equation:

$$(1-EE^*)\left\{\left[(1-t^2)E_{,t}\right]_{,t} - \left[(1-x^2)E_{,x}\right]_{,x}\right\} = -2E^*\left\{(1-t^2)(E_{,t})^2 - (1-x^2)(E_{,x})^2\right\}, \tag{2}$$

and the function f satisfies a set of **linear**, first order, partial differential equations, whose driving terms are given by the derivatives of the function E:

$$\frac{x}{1-x^2}F_{,t} + \frac{t}{1-t^2}F_{,x} = -\frac{2}{(1-|E|^2)^2}(E_{,t}E^*_{,x} + E_{,x}E^*_{,t}),$$
$$2tF_{,t} + 2xF_{,x} = \frac{3}{1-t^2} + \frac{1}{1-x^2} - \frac{4}{(1-|E|^2)^2}\left[(1-t^2)|E_{,t}|^2 + (1-x^2)|E_{,x}|^2\right], \tag{3}$$

where

$$F = log\frac{f}{\sqrt{1-t^2}}.$$

Once we know a solution for the function E, the linear system for f can be solved by quadrature. Thus the heart of the problem is the solution of the equation for E. The function E is complex, therefore the physical problem we are solving possesses two degrees of freedom corresponding to the two states of polarization of the colliding waves.

A point should be stressed. The equation satisfied by E is already known in a different context as the Ernst equation. In fact, in 1968 Ernst[7,8] showed that if one is looking for stationary axisymmetric solutions of Einstein's equations, the fundamental components of the metric can be combined into a single complex function which satisfies the Ernst equation, indeed. In that case, of course, the function E will depend on the radial distance from the center, and on the polar angle θ. However the equation is, formally, the exact same equation (2) derived in the context of colliding waves. The remaining equations for the component of the metric equivalent to our function f, are similar in structure to the set of equations (3). Thus, there exists a formal analogy between the mathematical theory of colliding waves and the mathematical theory of black holes, since both stationary axisymmetric spacetimes and spacetimes with two spacelike Killing vectors admit the same Ernst equation. Now, the study of the gravitational field of massive bodies is the natural context in which the validity of a theory of gravity must be tested. Therefore, since the general theory of relativity was formulated in 1916, stationary axisymmetric solutions have been extensively studied, and an enormous amount of work has been done in elaborating techniques and methods for solving these problems. Due to the analogy described above, it is clear that we can use all of the methods developed for stationary axisymmetric solutions over the past sixty years, to find solutions for the collision of gravitational waves. I will say nothing more about these techniques (for an extensive review see, for example, Ref. 9.) I shall now describe the general features of the wave-wave interaction process, as they emerge from the variety of solutions that have been found. Before starting, we need to remark upon the following fact. Let us introduce a couple of null coordinates $u = t - x$ and $v = t + x$. By the use of the aforementioned techniques, we are able to find the solution in the region where two waves interact, corresponding to region I in the two-dimensional diagram represented in Fig. 1.

Then, by using an algorithm introduced by Penrose, we extend the solution in the precollision regions II, III, and IV. The algorithm consists in assuming that in the regions before the collision, where respectively only an outgoing wave (II) or an ingoing wave (III) are present, the metric depends on u or on v only; that region IV, representing the spacetime between the two incoming waves, is flat, and that

$$\begin{aligned} g^{II}_{\mu\nu}(u) &= g^{I}_{\mu\nu}(u, v = 0)\,, \\ g^{III}_{\mu\nu}(v) &= g^{I}_{\mu\nu}(u = 0, v)\,, \\ g^{IV}_{\mu\nu} &= g^{I}_{\mu\nu}(u = 0, v = 0)\,. \end{aligned} \tag{4}$$

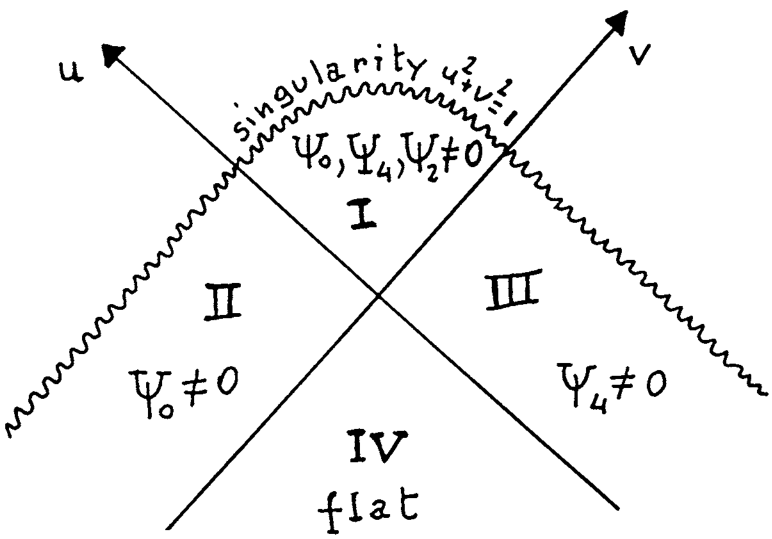

Fig. 1 The spacetime resulting from the collision of plane gravitational waves.

The requirements (4) are accomplished by the formal substitution

$$u \to uH(H)\,, \qquad\qquad v \to vH(v)$$

(where H is the Heaviside step function) in the expression of the metric valid in the interaction region.

The procedure we use for finding the complete solution is certainly unusual. We proceed, in some sense, as shrimps: find the solution in the region of interaction, then extend it back to the past to know what is the profile of the waves whose collision produced that interaction. However, even if we use an approach which could appear as exotic, the problem is well posed in the sense that the initial data on the null boundaries $u = 0$ and $v = 0$ uniquely determine the solution in the region of interaction (see Ref. 4). With the clarification of these points, we are now in a position to describe the common features of these solutions.

We can investigate their behavior by using the Weyl scalars, which provide essential information on the nature of the free part of the gravitational field. They are five complex scalars, constructed by projecting the Weyl tensor onto a suitable chosen null tetrad, and each of them carries particular information. We shall restrict ourselves to those scalars relevant to our problem, namely Ψ_0, Ψ_4, and Ψ_2.

Ψ_0 and Ψ_4 represent, respectively, the ingoing and the outgoing pure transverse radiative part of the field. This means that if we consider a ring of test particles, the forces generated by a field in which only Ψ_0 (or Ψ_4) is different from zero, will deform the ring into an ellipse. (We are assuming that the direction of the wave is orthogonal to the ring; the polarization axis will coincide with the axis of the ellipse.) If both polarizations are present, Ψ_0 (or Ψ_4) is complex and the deformation is the result of that described above including the same effect, however with the polarization axis tilted at 45^0. For a solution describing a single outgoing or ingoing gravitational wave (Petrov type N solutions) only Ψ_0 or Ψ_4 are different from zero.

Ψ_2 represents to so-called Coulomb-like part of the field. In fact, if $\Psi_2 \neq 0$, the corresponding gravitational force distorts a sphere of test particles into an ellipsoid. This is typical of the behavior of particles falling in toward a central attracting body with the inverse square law. For example, in the Schwarzschild and in the Kerr solution (or in general for Petrov type D solutions) only Ψ_2 is $\neq 0$. This beautiful description of the nature of a gravitational field in terms of the Weyl scalars was given by Szekeres in 1965[10] by analysing the equations of geodesic deviation.

We find the following situation when two gravitational waves collide. In the region before the collision respectively, only Ψ_0 and Ψ_4 are different from zero, due to the presence of an ingoing and an outgoing wave (see Fig. 1). In the region of interaction we still expect some mixture of ingoing and outgoing radiation and in fact $\Psi_0 \neq 0$ and $\Psi_4 \neq 0$, but in addition, we find that a Coulomb-like component of the field develops, since Ψ_2 turns out to be different from zero. Thus, the two waves do not pass through one another, namely, they do not superimpose and the non-linearity of the interaction manifests itself in the appearance of the Coulomb-like part of the field. This is a true interaction term. We can also say something more by analyzing the structure of the Weyl scalars. Let us do this by considering the simplest case, the Khan-Penrose solution, which describes the collision of two impulsive gravitational waves with collinear polarizations. Although this solution is simple, it exhibits most of the typical features present in more complicated solutions.

Imagine, for example, that the amplitude of both impulsive waves is A =const. Before the collision we have:

$$\Psi_0 = A\delta(u) \qquad \text{in region II},$$
$$\Psi_4 = A\delta(v) \qquad \text{in region III}.$$

After the collision, in the region of interaction:[11]

$$\Psi_0 = g(u)\delta(v) + k(u,v),$$
$$\Psi_4 = g(v)\delta(u) + k(v,u),$$
$$\Psi_2 = s(u,v)\theta(u)\theta(v).$$

From these expressions we deduce that the two impulsive waves continue after the collision, but with the amplitude scaled by the function g. In addition, they develop a tail given by the function $k(u,v)$, and it is interesting to note that k is a function of u and v in Ψ_0, and of v and u in Ψ_4. The conclusion is that part of the waves are transmitted in the region of interaction, part are reflected by each other and part of the incoming radiation transforms into a Coulomb-like gravitational field.

However, the most remarkable consequence of the interaction is the following: at a finite time from the instant of collision, and at a finite distance from the surface where the collision takes place, a curvature singularity appears, and the Riemann tensor diverges. The singularity is a spacelike singularity and it occurs on the hypersurface

$$u^2 + v^2 = 1.$$

In addition, it is a global singularity, because any test particle which is invested by one of the two waves, is forced to enter in the region of interaction, and to reach the singularity in a finite interval of proper time: nothing can escape the fate of being terminated onto the spacelike singularity.

It should also be stressed that the amplitude of the colliding waves in no way affects the creation of the singularity: even if the amplitude is very small, the singularity, sooner or later, appears. However the amplitude of the colliding waves, together with the other physical parameter at our disposal, the angle α between the two directions of polarization, determine the timescale for the creation of the singularity, according to the equation:

$$\Delta t = \frac{1}{A^2}\sqrt{1+\cos^2\alpha}\,, \qquad (5)$$

where Δt is the time interval between the instant of collision and the formation of the singularity, and A is the amplitude of the colliding waves.[12] Thus the time required in creating the singularity is inversely proportional to the intensity of the wave. A further delay is introduced if the two waves have non-collinear polarization.

However, the collision of gravitational plane waves does not necessarily produce a singularity as a final result. In fact there are solutions in which the physical singularity on the surface $u^2+v^2=1$ is replaced by a coordinate singularity and precisely, by a Killing-Cauchy horizon, similar to the horizon appearing in black hole solutions. In the black hole case the horizon is a compact hypersurface surrounding the singularity, while in the case of colliding waves it is a non-compact hypersurface. However, it possesses all the features which characterize a horizon, according to the standard definition: it is a smooth, null hypersurface on which the vector, which becomes null, is a Killing vector, and it is a one-way membrane.

The first solutions which were found to exhibit this non-singular behavior are described in Refs. 13-16. They present an interesting property: the region of interaction is isometric, respectively to a part of the Kerr, of the Schwarzschild, and of the Taub-Nut solutions inside the horizon. The isometry is only local since the Killing vectors have open orbits in our case and closed orbits in the case of black holes. It is interesting to note that in these solutions, the only nonvanishing Weyl scalar in the region of interaction is Ψ_2 and this means that the incident radiation completely transforms into a Coulomb-like field. The second interesting information is that, since the hypersurface $u^2 + v^2 = 1$ is only a coordinate singularity, these solutions can be extended across the horizon in a way which is similar to the Kruskal extension for the Schwarzschild metric or the equivalent extension for the Kerr metric. When the extension is performed, a singularity always appears. However, it can be a different type of singularity, for example, timelike in the solution found by Chandrasekhar and Xanthopoulos in Ref. 13. Finally, it should be noted that due to the presence of impulsive waves, these solutions always present a singularity at the points $u = 1$, $v = 0$, and $u = 0$, $v = 1$.[17] Other horizon-like solutions have been found, or analyzed, in Refs. 18 and 19. The last remark I would like to make on these solutions concerns their stability. Let us consider for example the solutions which are isometric to the Schwarzschild and the Taub-Nut metrics. They belong to a large class of soliton solutions identified by a set of parameters, and they correspond to a particular choice of those parameters. If one of the parameters is slightly changed, even by an infinitesimal amount, the solution immediately becomes singular. The choice of a particular set of parameters corresponds to a particular choice of the initial data on the null boundaries. Therefore, a perturbation of the initial data (or equivalently, a plane symmetric perturbation of these solutions) transforms the horizon into the usual spacelike singularity. In addition, Chandrasekhar and Xanthopoulos[20] have shown that the presence of an arbitrary small amount of null dust in the region of interaction would immediately change the horizon into a curvature singularity. These are clear indications of instability of the Killing-Cauchy horizons. (The problem of the stability of horizon-like solutions against small perturbations has also been analyzed in Refs. 21 and 22.)

3. Creation of curvature singularities

Apart from some exceptions, we have seen that the final result of the collision of gravitational plane waves is the creation of a curvature singularity. Therefore, the next question to answer is: why does the collision of gravitational plane waves generate such an infinite gravitational field? This occurrence is certainly related to a process of mutual focusing of the two colliding waves. This can be understood, for example, by studying the behaviour of null geodesics in the field of a single wave. If we consider a tube of neighboring null geodesics and take and infinitesimal ring orthogonal to them, compute how this ring expands or contracts, how it