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Topics in metric fixed point theory

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Preface

The term 'Metric' Fixed Point Theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and/or mappings play a crucial role. Obviously there can be no clear line separating this branch of fixed point theory from either the topological or set-theoretic branches since metric methods are often useful in proving results which are basically nonmetric in nature, and vice versa. However, the results considered here are always couched in at least a metric space framework, usually in a Banach space setting, and the methods typically involve both the topological and the geometric structure of the space in conjunction with metric constraints on the behavior of the mappings.

For the past twenty-five years metric fixed point theory has been a flourishing area of research for many mathematicians. Although a substantial number of definitive results have now been discovered, a few questions lying at the heart of the theory remain open and there are many unanswered questions regarding the limits to which the theory may be extended. Some of these questions are merely tantalizing while others suggest substantial new avenues of research.

It is apparent that the theory has now reached a level of maturity appropriate to an examination of its central themes. The topics selected for this text were chosen accordingly. No attempt has been made to explore all aspects of the theory nor to present a compendium of known facts. Our objective is merely to offer the mathematical community an accessible self-contained document which can be used as an introduction to the subject and its development. We have attempted to render the major results understandable to a wide audience, including nonspecialists, and at the same time to provide a source for examples, references, open questions, and (occasionally) new approaches for those currently working in the subject. The results presented in detail were selected to illustrate the directions research in this field has taken during the past twenty-five years and to illustrate the flavor of the subject. Also, in our attempt to render the topics treated self-contained, we only assume familiarity with the basic concepts of analysis and topology. To the extent these goals have been achieved, the text should be of interest to graduate students seeking a field of interest, to mathematicians interested in learning about the subject, and to specialists.

The structure of the text is straightforward. There are twenty-one short chapters devoted to various aspects of the theory. A substantial portion of the book, Chapters 3–13, is devoted to the classical theory of nonexpansive mappings. Included in these chapters is a discussion of the basic problems of the field: the existence of fixed points, the structure of the fixed point sets, and approximation techniques for locating fixed points. Chapter 14, Ultrafilter Methods, is exceptional in that it contains some recent results obtained by utilizing 'nonstandard' techniques based on the concepts of ultrafilters, ultrapowers and ultranets. These methods are nonintuitive and not usually viewed as 'metric geometry' tools. Nevertheless, they are powerful techniques which seem capable of laying the foundation of an entirely separate branch of the subject. Since results in this direction are still emerging, we provide only an introduction to the subject here.

Chapters 14, 15 and 16 are devoted to some generalizations and extensions of the previous results to classes of mappings which are not necessarily nonexpansive but which satisfy closely related metric constraints, and the last four chapters contain some relatively fresh problems of metrical type which evolved from the classical theorems of Brouwer and Schauder (or, more precisely, from 'naive' attempts to obtain generalizations of these results to noncompact settings).

The theory treated here has many contributors. Those who developed the classical theory include the celebrated mathematicians L. E. J. Brouwer, S. Banach, and J. Schauder. The metric theory was given a new impetus by the 1965 fixed point theorems for nonexpansive mappings discovered independently by F. Browder, D. Göhde, and W. A. Kirk, and by the widely circulated *1967 Lecture Notes* of Z. Opial. There have been numerous major discoveries since then, and this text is made possible only because of those contributions as well as the contributions of many pioneering mathematicians who developed the functional analytic framework within which most of the subject is couched.

We thank collectively our many friends and colleagues who, through their encouragement and help, influenced the development of this book. We are particularly grateful to Juan Gatica, who examined portions of the manuscript in detail, and to Stanislaw Prus, whose astute observations led to significant improvements in Chapter 9. And we especially thank Tadeusz Kuczumow for pointing out numerous oversights in the original draft of the manuscript. Finally, we thank our typist, Julie Hill, for skilfully and patiently seeing the manuscript through its various stages.