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EDITED BY G.-C. ROTA

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Exact constants in approximation theory

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Exact constants in approximation theory

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PREFACE

This book was written for the series Encyclopedia of Mathematics and Its Applications published in the USA and United Kingdom since 1974. A few years ago I was asked to write a monograph *Exact Constants in Approximation Theory* by the editor of the series Professor G.-C. Rota.

The purpose and objectives of the series were presented by the editorial board as their credo in the following statement:

> It was not too long ago that the view that in present-day mathematics one can find the basic concepts of tomorrow's science was widely spread. Nowadays, however, mathematical results, proudly secured behind the barrier of complicated terminology with uncompromising strictness, are very often far removed from their potential users . . . The ample results along with sophisticated modern mathematical presentation have resulted in the mathematician's reluctance to view his achievements from aside, with the eyes of an interested applied scientist. This fact and our conviction that every science should sooner or later test itself by becoming mathematical are the motivating force of the Encyclopedia.

In their letter to authors the editorial board gives them a free choice of material and style of presentation imposing only two requirements that the monograph should meet, namely: (1) the content should not be of transient significance; and (2) the form of presentation should make the subject matter comprehensible for a wide circle of reades, including non-specialists who may be dealing with other branches of mathematics.

In our view a sign of the healthy development of every theory is the fact that at a given stage it is used in adjoining or applied fields of science. Approximation theory like some other theoretical branches of mathematics, having originated from practical problems, was for a while mainly with its own internal problems. At present, however, it can be observed that the ideas and methods of approximation theory have spread to various fields of the natural sciences to a much higher extent than, let us say, twenty or fifty years ago. The same is true for the successful application of its achievements in applied research. The major successes in the solution of extremal problems (finding N-widths, in particular) which have served as a theoretical foundation in investigating many practical optimization problems were of high importance.

The choice of material was predetermined by the title of the book, that is it had to include results in which the exact constants in approximation problems are found. Thus, the question of the subject matter was more or less clear from the very beginning, whereas the question of the form of presentation proved to be much more difficult for the author. This was not only because of the necessity of making the monograph comprehensible for non-specialists. It would not have been that difficult to present the results for exact constants in a systematic way supplementing them with some clarifications. But that was not what I wanted. To get to the essence of the method of solution is much more important than to know the final results, not only for the pure scientist but also for the applied scientist who is looking for mathematical approaches to the solution of practical problems for which the standard schemes do not work. Furthermore quoting from the editors' statement again, they say 'what is most important in a theory is not always contained in the formulations of the theorems, it can be found in the proof's arguments, in algorithms, examples and even in drawings'. This applies to the present book because in obtaining the exact constants in an approximation problem a new approach usually emerges based on entirely new ingredients (e.g. a new exact inequality) often bearing a simple geometrical meaning. It is frequently found that this new approach also paves the way to the solution of other often quite different problems.

Moreover, not every mathematically precise proof should be included in a book written for a wide circle of readers. Sometimes it takes time to determine both the exact place of the new result in the general system of mathematical facts and the chain of arguments leading to it in the shortest and most natural way. In the long run both the result and its proof find their places. It is such proofs which combine depth and simplicity that bring aesthetic pleasure and allow us to speak of beauty in mathematics. In some sense they are exact constants of a kind because they cannot be further improved. I do not mean to claim that the proofs in the present book meet those high requirements though I am not hiding my aspirations towards such an ideal.

The most important key results as well as the statements clarifying the essence of the method are supplemented by detailed proofs. We would like to point out that the methods for solving the extremal problems used in this book are based, as a rule, on geometrical considerations and quite often a fact, whose analytical description is complicated, becomes clear when a drawing is made.

One can judge the contents of each chapter by its title. I shall say only a few words about the general principles of my presentation. The exact constants results in approximation problems, as we presently understand it, belong, in most cases, to approximation by polynomials (trigonometric or algebraic) and by polynomial splines; Chapters 4 to 7 are devoted to these questions. Some sections (e.g. 4.3 and 4.4) and results can be studied independently of the preceding material. The majority of the exact constants related, in particular, to the best approximation are based on deep facts of analysis and function theory, therefore, I considered it expedient to present this foundation material in Chapter 1 (duality relations) and Chapter 3 (comparison theorems). As a matter of fact, exact constants inequalities proved in Chapter 3 belong to the general subject matter of the book by their very nature. Chapter 2 expresses both my intention for independent presentation and my wish to avoid the explanation and substantiation of some general facts from approximation theory (actually well known to specialists) later on. Chapter 8 gives the logical completion of the main lines of solving extremal problems by determining the values of N-widths of some classes of functions of finite smoothness. Furthermore, it is found that the exactness of the constants in most of the results in Chapters 4 to 7 can be interpreted in a much wider sense.

In attempting to achieve an optimal volume for the book we have, of course, not been able to give all aspects of modern approximation theory even from the exact constants point of view. Some material from optimal quadratures theory, although very rich in exact results, remains beyond the scope of the present book. Not enough light has been shed on optimal recovery of functions and linear functionals, approximation of convolution functional classes either.

The references are given in three lists. Lists A and B contain monographs in approximation theory and books of a more general or other nature, respectively. We quote them as [1A], [1B], etc. The third list is of articles and other original publications in approximation theory which are directly related to the subject of the book.

The author would like to thank the referee V. M. Tikhomirov for his support in publishing the book and his recommendations for its improvement, to V. F. Babenko whose remarks on the manuscript enable me to improve some parts of the presentation and to Zh. E. Myrzanov for his assistance in preparing the manuscript for printing.

N. Korneichuk

LIST OF MOST IMPORTANT NOTATION

Notation of general nature

$\forall$	generality quantor: 'for all'
$\exists$	existence quantor: 'there is'
$\mathbb{R}$	the set of real numbers
$\varnothing$	the empty set
$x \in A$	the element x belongs to the set A
$x \notin A$	the element x does not belong to the set A
$A \cup B$	the union of sets A and B
$A \cap B$	the intersection of sets A and B
$A \backslash B$	the difference of sets A and B
$A \oplus B$	the direct sum of sets A and B
$A \subset B$	the set A is contained in the set B
$\underline{A}$	the closure of set A
$\{x : P_x\}$	the set of all elements x possessing property P
$\sup\limits_{x \in A} f(x)$ (or $\sup \{f(x) : x \in A\}$	the supremum of the values of functional f on set A
$\inf\limits_{x \in A} f(x)$ (or $\inf \{f(x) : x \in A\}$	the infimum of the values of functional f on set A
$=:$	equal by definition
$\operatorname{sgn} \alpha$	a quantity equal to 1 if $\alpha > 0$, equal to -1 if $\alpha < 0$, and zero if $\alpha = 0$
ess sup	essential supremum
meas E	the Lebesgue measure of set E
dim X	the dimension of linear space X

$f \perp \varphi$ for $f \in L_1[a, b]$ and $\varphi \in L_\infty[a, b]$ means $\int_a^b f(t)\varphi(t)\, \mathrm{d}t = 0$

$f \perp \mathcal{N}$ for $f \in L_1[a, b]$ and $\mathcal{N} \subset L_\infty[a, b]$ means $f \perp \varphi \ \forall \varphi \in \mathcal{N}$

$[\alpha]$	the integral part of real number α
δ_{ki}	Kronecker symbol: $\delta_{ii} = 1$, $\delta_{ki} = 0$ $(k \neq i)$