Yet another introduction to analysis

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1

Firm foundations

Where do we start?

Analysis is an extension of school calculus. By building on firm foundations we intend to develop a rigorous study of the behaviour of functions. To start you off, then, try the following exercise:

$$f(x) = x^2 \quad x \in \mathbb{R}$$

$$\text{Calculate} \quad f'(1) \quad \text{and} \quad \int_1^2 f(x)\,\mathrm{d}x$$

I hope that you found that easy: $f'(1)$ (or $\mathrm{d}f/\mathrm{d}x$ at $x = 1$) is 2 and the integral is $2\frac{1}{3}$.

But what on earth do those answers mean? Aren't they something to do with gradients and area? Where did those techniques you used come from and why do they work? Try another exercise:

$$g(x) = |x| \quad x \in \mathbb{R}$$

$$\text{Calculate} \quad g'(-3) \quad \text{and} \quad \int_{-1}^2 g(x)\,\mathrm{d}x$$

You may be a little puzzled by that example as the 'modulus' function, although simple enough, is not usually included in school calculus. Never mind, consider this next example. Another function with a lot of practical uses is 'the integer part' function, where $[x]$ denotes the *integer part* of x, for example $[3.27] = 3$. Try this third exercise:

$$h(x) = [x] \quad x \in \mathbb{R}$$

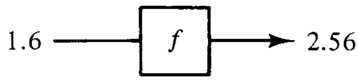
$$\text{Calculate} \quad h'(2) \quad \text{and} \quad \int_0^3 h(x)\,\mathrm{d}x$$

I suspect that most of you will be giving up by now or (hopefully) beginning to think a little more about what these things mean.

Our only hope of building a solid study of analysis is by starting from firm foundations, so there's no point in plunging straight into the calculus. Let us go back to the idea of a function:

$$f(x) = x^2 \quad x \in \mathbb{R}$$
$$g(x) = |x| \quad x \in \mathbb{R}$$
$$h(x) = [x] \quad x \in \mathbb{R}$$

A function takes in a number and gives out another. For example



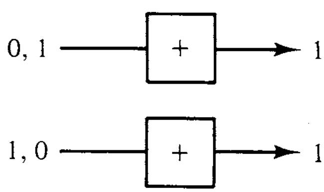
So surely to understand how functions behave we must first understand the behaviour of the numbers themselves? Our very first mathematical-looking statement above was

$$f(x) = x^2 \quad x \in \mathbb{R}$$

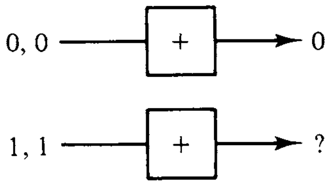
But what *is* $\mathbb{R}$? Our study of analysis surely *has* to start with a study of numbers.

A fresh beginning

We've now decided on a fresh start: from this point onwards we must make clear what our assumptions are and base all our deductions upon those assumptions. But how far back should I go? I could assume that you have absolutely no knowledge of numbers and begin by naming two new creations, '0' and '1', the first 'numbers'. I could then introduce an operation, '+', which takes in any pair of existing numbers and gives out a number as an answer. Since our only numbers so far are 0 and 1 we try those:



So far that is very boring indeed. But the pair of numbers fed in don't have to be different so we try:



and the last example will give us a new number which we might call '2'. Then 2 + 1 would give us a number which we might call '3', and so on. This would be a laborious way of introducing the number system. It would be rather like giving a history course which started with the formation of the first algae in the oceans: by the time you got to any interesting historical events the audience would have died from boredom. So such a basic beginning would be inappropriate here. Like the historian who, for the sake of an interesting and complete course, chooses a convenient starting-point such as the outbreak of the first world war, we are going to take a giant leap forward:

We shall assume that all the arithmetic which you met at primary school works in exactly the way you'd expect it to.

We therefore assume that there are whole numbers (or *integers*)

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

(the set of all those is referred to as $\mathbb{Z}$) upon which the normal operations of addition, subtraction and multiplication can be applied, giving answers back in the same set. These integers have a natural order and given any two different integers, m and n say, one of them will be less than the other. If $m < n$ for example then starting at m and 'counting' m, $m + 1$, $m + 2, \ldots$ we will eventually get to n.

We also assume that any integer can be divided by any non-zero integer to give a fractional (or *rational*) number. For example,

$$2\tfrac{1}{2}\ (= 5 \div 2 = 10 \div 4 \text{ etc}),\ 3\tfrac{1}{7},\ -22\tfrac{7}{8}, \ldots$$

and the set of all such rational numbers is denoted by $\mathbb{Q}$. These rational numbers again behave exactly as you'd expect them to from school. Any rational number m/n can be 'cancelled down' to its lowest form (for example 84/126 would cancel down to 2/3).

Furthermore the four arithmetic operations extend in a natural way to $\mathbb{Q}$ and behave exactly as you'd expect them to. For example,

$$3\tfrac{1}{2} + 7 = 10\tfrac{1}{2} = 7 + 3\tfrac{1}{2}$$

– the order does not matter. In general

$$a + b = b + a$$

for each of our numbers a and b (and we say that $+$ is *commutative*). Similarly

$$\underbrace{(2 + 5\tfrac{3}{4})}_{\substack{\text{done}\\\text{first}}} + 4\tfrac{1}{2} = 12\tfrac{1}{4} = 2 + \underbrace{(5\tfrac{3}{4} + 4\tfrac{1}{2})}_{\substack{\text{done}\\\text{first}}}$$

so that in general the sum $a + b + c$ makes sense without brackets, this property being called *associativity*. The same two rules hold for multiplication too. Also the two operations of multiplication and addition combine in natural ways such as with the rule

$$a(b + c) = ab + ac$$

called *distributivity*. All these rules taken together mean that brackets can be multiplied out and manipulated in the usual ways.

By making our assumptions we have taken a large leap forward and we have a viable working number system. But is it complete enough for us to be able to consider properly things like functions and differentiation etc, which is what analysis is all about? No, because even assuming all the arithmetic that we learnt at primary school there is still a huge gap in our number system.

Primary arithmetic is not enough to lead us to a full study of decimals, especially as some decimal expansions are infinite (they go on for ever) and it's not clear from primary arithmetic what that means. However, I'd now like to use decimals in an informal way to motivate where we go next. As long as we only use the decimals to motivate us then we are still keeping to our brief of basing our formal *development* on the primary arithmetic alone.

Informally then let us represent the rational numbers by their decimal expansions. For example,

$$\begin{aligned} 1 &= 1.0000\ldots \text{ (a 'finite' decimal, or one with 0s recurring)} \\ 1\tfrac{1}{2} &= 1.5000\ldots \text{ (again with 0s recurring)} \\ 2\tfrac{719}{1650} &= 2.43575757\ldots \text{ (with 57s recurring)} \\ 3\tfrac{1}{7} &= 3.142857142857\ldots \text{ (with 142857s recurring)} \end{aligned}$$

All these decimals (and indeed the decimal expansions of all the numbers in $\mathbb{Q}$) recur, that is they eventually have a repeating pattern. Of course this recurring pattern may simply be of zeros, which one doesn't bother writing. But surely it's easy to make up decimals which do not have a recurring pattern? One of the most famous examples is

$$\pi = 3.141592653589793\ldots$$

Yet our primary school construction of the arithmetic has completely missed such numbers. As a first step to incorporating them we now let $\mathbb{R}$ denote the set of all *real numbers* (which will turn out to be the set of all numbers with a decimal expansion). Whereas those numbers with recurring decimals were the rational numbers, all the rest will be called *irrational*.

We won't be sure about the existence and behaviour of the irrational numbers until we have decided which assumption to make in addition to our assumption about primary school arithmetic. But just in case you should think that the irrationals are very few-and-far-between and that they are not worth worrying too much about, try answering the following informal question:

> **Question** Do you think that there are more rationals than irrationals, more irrationals than rationals, or about the same number of each?

Counting such huge collections is perhaps a little dodgy, so you might prefer the question in an alternative form:

> **Question** I choose a real number between 0 and 1 at random. (For example I could have a die with ten faces and the digits 0–9 on those faces. I could then choose the decimal expansion randomly by repeatedly throwing the die.) Would the resulting number be more likely to be rational (with a constantly repeating pattern) or irrational (with no repeating pattern)?

Surely the latter is far more likely, so much so that one would hardly ever choose a rational number. So, far from being a minor gap in the real numbers, the irrational numbers will form the major part of $\mathbb{R}$.

We now pause for some exercises. In general these will form a crucial part of our development and you are urged to give them careful thought before referring to the solutions on page 221.

Exercises

1 By representing two typical rational numbers by m/n and p/q, where m, n, p and q are integers, show that the sum, difference, product and quotient of two rational numbers are all rational numbers. (Of course we shall always assume that we will only divide one number by another when the latter one is non-zero.)

Deduce that the sum and difference of a rational number r with an irrational number s gives irrational answers and that the product and quotient of a non-zero rational number r with an irrational number s gives irrational answers.

2 In one of the above examples $3\frac{1}{7} = 3.142857142857\ldots$, with a repeating string of six digits. Show why in the decimal expansion of m/n (where m and n are integers and $n > 0$) there is eventually bound to be a recurring 0 or a recurring string of less than n digits.

3 It's a surprising fact that given any positive integer n some multiple of it is of the form $99\ldots9900\ldots00$. For example given the number 74 it turns out that $135 \times 74 = 9990$. (Perhaps this is connected with the fact that

$$\tfrac{1}{74} = 0.0135135135135\ldots)$$

Prove the general result.

(Solutions on page 221).

A not-so-simple equation

Even with your basic primary school arithmetic you were soon able to move into the world of algebra and solve simple equations like

$$3 + x = 5$$

$$7x = 11$$

because in each case a straightforward arithmetic operation yields the solution. But what about the equation

$$x^2 = 2$$

How do we know that there exists an x satisfying this equation? The idea of taking square roots of 2 makes no sense if we are only allowed to assume primary school arithmetic. You could of course use your calculator and tell me that the answer is 1.414 214, but that's only an approximation since

$$(1.414214)^2 = 2.000001237796$$

In fact, since a calculator can only display a finite number of decimal places, all its answers are rational numbers. With more and more powerful calculators you might be able to obtain better and better approximations to a solution of $x^2 = 2$: e.g.

$$1.414214,\ 1.4142136,\ 1.41421357,\ldots$$

all of which give squares larger than (but closer and closer to) 2. But by this method you'll never find a number whose square *equals* 2. In fact, if you restrict attention to rational numbers (which is all you're allowed from your primary school maths) then there is no x which satisfies $x^2 = 2$, as we now see in our first 'theorem'; i.e. a result worth highlighting for future reference:

Theorem There is no rational number x with $x^2 = 2$.

(Having stated that result we now need to prove it using only the assumptions stated earlier. Proofs of future theorems can in addition use results already established.)

Proof Let $x = m/n$ where m and n are integers with $n > 0$. We shall take for granted the fact that the fraction m/n is written in a sensible form, with no factor in both m and n which could be cancelled down (for example we wouldn't write 38/24 when 19/12 would do).

We'll now assume that $x^2 = 2$ and we'll deduce a piece of nonsense, showing that our assumption is wrong:

$$\frac{m^2}{n^2} = \left(\frac{m}{n}\right)^2 = x^2 = 2$$

$$\therefore\ m^2 = 2n^2$$

$$\therefore\ m^2 \text{ is even}$$

But the squares of odd numbers are themselves odd. So how can the square of an integer be even? The integer itself must be even. But then its square is divisible by 4 i.e.

$$\boldsymbol{m} \textbf{ is even}$$

$$\therefore\ m^2 = 2n^2 \text{ is divisible by 4}$$

$$\therefore\ n^2 \text{ is even}$$

$$\therefore\ \boldsymbol{n} \textbf{ is even}$$

But this means that both m and n are even, contradicting the fact that the fraction m/n was written in a sensible form.

Hence if we assume that the rational number x has $x^2 = 2$, then we get a contradiction, so there's no such rational. □

Actually there's a very swish alternative way of seeing that there is no rational number which gives a sensible value of '$\sqrt{2}$' (or of '$\sqrt{q}$' unless q is a perfect square) which uses properties of prime factors. For example imagine that you thought that the rational number $x = 1.41242424\ldots$ gave $x^2 = 2$. You could then write x as a cancelled-down fraction and then express the top and bottom of that fraction as a product of primes; i.e.

$$x = 1.412424\ldots = \frac{4661}{330} = \frac{59 \times 79}{2 \times 2 \times 5 \times 5 \times 3 \times 11}$$

But then

$$2 = 1.412424\ldots^2$$
$$= \frac{59 \times 79 \times 59 \times 79}{2 \times 2 \times 5 \times 5 \times 3 \times 11 \times 2 \times 2 \times 5 \times 5 \times 3 \times 11}$$

and that's clearly impossible since *still* nothing cancels on the right-hand side. This method relies on prime factorisation and a proof based on this idea will be found in the next exercises.

So we now know that there is no rational number x with $x^2 = 2$, but how can we be sure that there *is* such an irrational number? Our primary arithmetic is no help to us when it comes to irrational numbers. In order to complete our study of $\mathbb{R}$ we need one further fundamental assumption, but it's not easy to see what form that assumption should take. We'll investigate that further in the next section.

Exercises

1 Show that $\sqrt{3}$ and $\sqrt[3]{2}$ are not rational.

2 Are you happy to accept that an eventual conclusion from primary school arithmetic is that every integer larger than 1 can be uniquely expressed as a product of prime numbers? (For example

$$1320 = 2 \times 3 \times 2 \times 5 \times 11 \times 2$$

where obviously the order of the factors is irrelevant.) If so, use this fact to show that if m and n are positive integers then it is impossible to have $m^2 = 2 \times n^2$. (Assume that m is the product of M primes, and n the product of N primes, and see what conclusion you come to.) This gives an alternative proof that $\sqrt{2}$ cannot be rational. If you're keen, show that if q is a positive integer then the only way that $\sqrt{q}$ can be rational is when q is a perfect square, in which case $\sqrt{q}$ is itself an integer.

3 (i) Suppose you are given a positive number α with $\alpha^2 > 2$. Then let

$$\beta = \frac{\alpha}{2} + \frac{1}{\alpha}$$

Show that $0 < \beta < \alpha$ and that $\beta^2 > 2$. Explain why this shows that there is no smallest positive number whose square is more than 2.

(ii) Suppose now that you are given a positive number α with $\alpha^2 < 2$. By considering $2/\alpha$ and using (i), show that there is a number β with $\beta > \alpha$ and $\beta^2 < 2$. Deduce that there is no biggest number whose square is less than 2.

(Solutions on page 223)

Piggy-in-the-middle

You will probably have met a little set theory at school: a *set* is simply a collection of objects, and for our purposes these objects will always be real numbers. In other words, all the sets which we consider are subsets of $\mathbb{R}$. Examples are

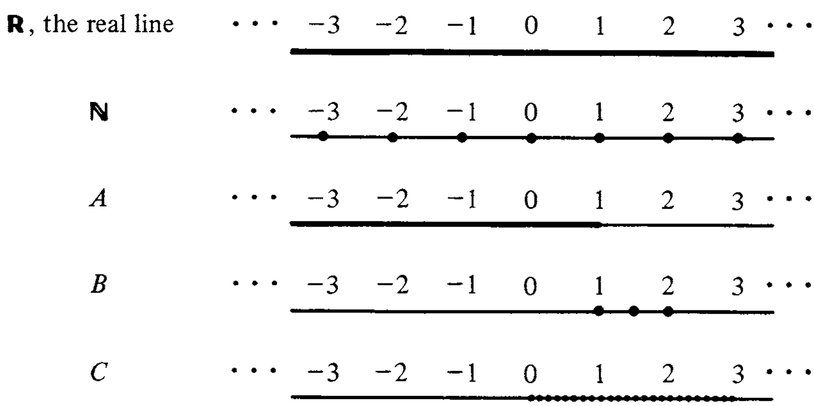
$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ – the set of *natural numbers*

$A = \{x \in \mathbb{R}: x < 1\}$

$B = \{1, 1\frac{1}{2}, 2\}$

$C = \{x \in \mathbb{Q}: 0 \leqslant x \leqslant 3\}$

The last example can be read as 'those x in $\mathbb{Q}$ – i.e. those rational x – which satisfy $0 \leqslant x \leqslant 3$'. Examples of *members* or *elements* of C are 1.5 and 2.32: we write $1.5 \in C$. It is sometimes convenient to visualise a subset of $\mathbb{R}$ informally by thinking of $\mathbb{R}$ as an infinite ruler and by marking that subset as part of the ruler. The following pictures illustrate $\mathbb{R}$ and the four sets $\mathbb{N}$, A, B and C defined above:



Some special sets which we shall encounter are called *intervals*: these are sets with the property that if two numbers are in the set then so are all the numbers between them. So the only intervals in the above examples are $\mathbb{R}$ itself and the set A. The set B, for example, is not an interval because it contains 1 and $1\frac{1}{2}$ but not the number $1\frac{3}{8}$ which is between them. Similarly C fails to be an interval because 0 and 3 are in C but, as we shall soon see, there exists an irrational number between them which is therefore not in C.

In general we write

$[a, b]$ for the 'closed' interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$

(a, b) for the 'open' interval $\{x \in \mathbb{R}: a < x < b\}$

$(a, b]$ for the interval $\{x \in \mathbb{R}: a < x \leqslant b\}$

$(-\infty, a)$ for the interval $\{x \in \mathbb{R}: x < a\}$

$[a, \infty)$ for the interval $\{x \in \mathbb{R}: x \geqslant a\}$ etc.

Earlier we started to consider some positive real numbers whose squares are More than 2. Now let M be the set of *all* such numbers; i.e.

$$M = \{x \in \mathbb{R}: x > 0 \text{ and } x^2 > 2\}$$

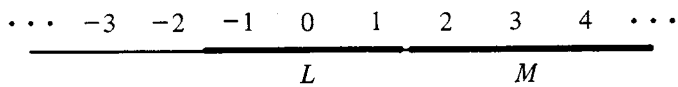
Then, as we saw earlier, examples of members of M are

1.414214, 1.4142136, 1.41421357, ...

We could, in a similar way, let L be the set of all real numbers whose squares are Less than 2; i.e.

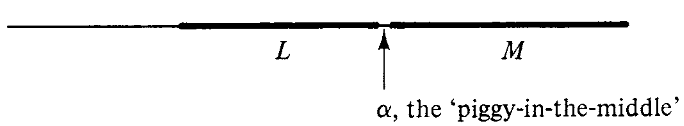
$$L = \{x \in \mathbb{R}: x^2 < 2\}$$

Examples of members of L are 1.414213, 1.4142135 etc.:



The picture of L and M shows their relative position: by sticking to positive numbers in M, L seems to be 'less than' M in some sense. And where do we expect $\sqrt{2}$ to be in relation to the two sets L and M? We would surely expect it to be 'between' them. There is no way that the existence of a number 'between' L and M can be deduced from primary arithmetic alone and so in order to proceed we need a further assumption:

If L and M are non-empty sets with $l \leqslant m$ for each $l \in L$ and each $m \in M$, then there exists a real number α such that $\alpha \geqslant l$ for each $l \in L$ and $\alpha \leqslant m$ for each $m \in M$.



This extra assumption is known as the *completeness axiom*: it ensures that there are 'no gaps' in $\mathbb{R}$. All analysis texts have to make an equivalent assumption somewhere and later we shall derive an equivalent form of this axiom found in many books. To see what the completeness axiom means in practice consider the following examples:

(1) $L = \{x \in \mathbb{R}: x \leqslant 2\}$ $\quad M = \{x \in \mathbb{R}: x \geqslant 2\}$ $\quad \alpha = 2$

In general if the non-empty sets L and M satisfy $l \leqslant m$ for each $l \in L$ and

$m \in M$ then they can have at most one number in common. If they do have a number in common then that number is the α of the axiom.

(2) $L = \{x \in \mathbb{R}: x < 1\}$ $\qquad M = \{4, 4\frac{1}{2}, 5, 5\frac{1}{2}, 6, \ldots\}$

In cases like this there are many choices for α; here any α in the interval $[1, 4]$ will do.

(3) $L = \{x \in \mathbb{R}: x^2 < 2\}$ $\qquad M = \{x \in \mathbb{R}: x > 0 \text{ and } x^2 > 2\}$

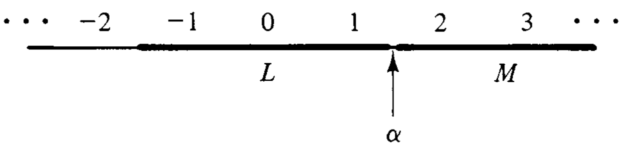
It is examples like these (which we met above) for which the completeness axiom is indispensible. In this case it ensures the existence of $\sqrt{2}$, as we now see:

Theorem $\sqrt{2}$ exists.

Proof As above let L and M be the sets

$$L = \{x \in \mathbb{R}: x^2 < 2\} \qquad M = \{x \in \mathbb{R}: x > 0 \text{ and } x^2 > 2\}$$

Then L and M are clearly non-empty (for example $1 \in L$ and $3 \in M$) and $l \leqslant m$ for each $l \in L$ and $m \in M$. Hence by the completeness axiom there is a number α with $\alpha \geqslant l$ for each $l \in L$ and $\alpha \leqslant m$ for each $m \in M$ (and in particular α must be positive).



In exercise 3 on page 8 we saw that there is no smallest positive number whose square is more than 2; i.e. M has no smallest member: hence α is not in M. **Thus $\alpha^2 \leqslant 2$.**

Similarly in that exercise we saw that there is no biggest number whose square is less than 2; i.e. L has no biggest member: hence α is not in L. **Thus $\alpha^2 \geqslant 2$.**

Hence α^2 equals 2 and we have found the number α equal to $\sqrt{2}$. □

In a similar way we can prove the existence of $x^{1/n}$, the nth root of x, for any non-negative number x and any positive integer n. Hence $x^{m/n}$ $(=(x^{1/n})^m)$ makes sense for the rational number m/n and, for the moment, we shall only refer to x^r when r is rational.

A common form of the completeness axiom concerns 'bounds' of sets. It will surely be immediately clear what we mean by a set being 'bounded above': in the four examples we met earlier

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$
$$A = \{x \in \mathbb{R} : x < 1\}$$
$$B = \{1, 1\tfrac{1}{2}, 2\}$$
$$C = \{x \in \mathbb{Q} : 0 \leqslant x \leqslant 3\}$$

the sets A, B and C are bounded above but $\mathbb{N}$ does not seem to be. (We shall verify that fact in the next exercises. Don't ever be tempted to think that ∞ is a number – what on earth would its decimal expansion be?) In general a set E is *bounded above* if there exists a number u (which will be called an *upper bound*) such that $e \leqslant u$ for all e in E. The set A given here is bounded above, and examples of its upper bounds are 4, 270 and $1\frac{1}{2}$ (there are lots of them! – we didn't say that u was unique) although perhaps the most natural choice of upper bound would be the number 1, being the smallest of all the possible upper bounds of A. The smallest upper bound of a set E is sometimes called the *supremum* of E, and is denoted by $\sup E$ (it's called *the* supremum because if there is one it *is* unique). In the four examples above the sets A, B and C were non-empty and bounded above. Their suprema (= plural of supremum) are given by

$$\sup A = 1 \qquad \sup B = 2 \qquad \sup C = 3$$

Can we now be sure, with the aid of our new assumption, that non-empty sets which are bounded above always have a smallest upper bound (or supremum)? We now prove that result, often used in text books in place of our completeness axiom:

Theorem If L is a non-empty set which is bounded above then L has a supremum.

Proof Let L be a non-empty set which is bounded above and let M be the set of upper bounds of L. Then as L has at least one upper bound it follows that the set M is non-empty. In addition it is clear that $l \leqslant m$ for each $l \in L$ and each $m \in M$.

We can now apply the completeness axiom to L and M to deduce that there exists a number α with $\alpha \geqslant l$ for each $l \in L$ (so that α is an *upper bound* of L) and $\alpha \leqslant m$ for each $m \in M$ (so that α is the *smallest* of all the upper bounds).

Hence α is the least upper bound (or supremum) of L. □

Of course there is nothing special about the top end of a set: we can define what we mean by a set being 'bounded below' and look for its largest lower bound and we can talk about a set being *bounded*, which means that it is both bounded above and bounded below.

Exercises

1 Given any positive number u let $[u]$ denote its 'integer part'. Show that $[u] + 1$ is a positive integer which is larger than u. Deduce that the set $\mathbb{N}$ of positive integers has no upper bound.

2 (i) Which of these sets are bounded above? What are their suprema?

$$A = \{\text{the prime numbers}\}$$

$$B = \{1/p\colon p \text{ is a prime number}\}$$

$$C = \{\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{6}{7}, \tfrac{7}{8}, \tfrac{8}{9}, \tfrac{9}{10}, \ldots\}$$

$$D = \{1, \tfrac{2}{3}, \tfrac{3}{5}, \tfrac{4}{7}, \tfrac{5}{9}, \tfrac{6}{11}, \tfrac{7}{13}, \tfrac{8}{15}, \ldots\}$$

(ii) Show that if A and B are any non-empty sets with $a + b \leqslant \alpha$ for each $a \in A$ and $b \in B$ then A and B are bounded above and

$$\sup A + \sup B \leqslant \alpha$$

3 (i) Show that a non-empty set M which is bounded below has a biggest lower bound (called the *infimum* and denoted by $\inf M$).

(ii) Which of the sets in exercise 2(i) are bounded below? What are their infima?

(iii) Now let L and M be any non-empty sets with $l \leqslant m$ for each $l \in L$ and $m \in M$ and such that there is a *unique* number α 'between' L and M (i.e. with $l \leqslant \alpha \leqslant m$ for each $l \in L$ and $m \in M$). Show that

$$\alpha = \sup L = \inf M$$

(iv) Let A be a non-empty set which is bounded below and let B be the set $\{-a\colon a \in A\}$. Show that B is bounded above and that

$$\sup B = -\inf A$$

4 (i) Let E be a set and consider the following two statements about E:

(1) there exists a number b with $-b \leqslant e \leqslant b$ for all $e \in E$;
(2) E is bounded (i.e. bounded above and bounded below).

Show that if (1) is true then so is (2), and that if (2) is true then so is (1). (Mathematicians say that (1) happens **if and only if** (2) does.)

(ii) Prove that the union of two bounded sets is bounded. (Remember that the union $E \cup E'$ of the sets E and E$'$ consists of all the elements which were in E or in E' or both.)

5 Let x and y be different real numbers and let $0 < \beta < 1$. Show that the number $x + \beta(y - x)$ is between x and y. By choosing appropriate values of β show that between any two different rational numbers there are both rational numbers and irrational numbers, and that between a rational and an irrational number there is an irrational number.

(We shall deduce in the next exercises that between any two different numbers there are both rationals and irrationals.)

6 One of the following school exercises is wrong: which one, and why?

(i) The attendance at a football match is 23 000 to the nearest thousand. What is the largest number of people that could have been at the match?

(ii) An angle is measured to the nearest degree and found to be 48°. What is the largest possible value of the angle?

(Solutions on page 224)

Some natural consequences

Sometimes a set may have a biggest or a smallest member, known as its *maximum* or *minimum*: note carefully that the maximum or minimum of a set has to be a member of the set. For example the set

$$\{x \in \mathbb{R}: 0 < x \leqslant 3\}$$

has a maximum of 3 but no minimum member (if you give me *any* number in the set then I can find a smaller one by halving your number), whereas the set

$$\{2, 4, 6, 8, 10, \ldots\}$$

has a minimum but no maximum. In exercise 6 above the set

$$\{n \in \mathbb{N}: n \text{ to the nearest thousand is } 23\,000\}$$

has a maximum member 23 499 but the set

$$\{x \in \mathbb{R}: x \text{ to the nearest whole number is } 48\}$$

had no maximum. It seems that bounded sets of integers are more likely to have maximum and minimum members than arbitrary sets of numbers, as we now establish.

> ***Theorem*** Any non-empty set of integers which is bounded above has a maximum member. Similarly, any non-empty set of integers which is bounded below has a minimum member. In particular any non-empty subset of the set $\mathbb{N}$ of positive integers has a minimum member.

Proof Let E be any non-empty set of integers which is bounded above. Then E contains some integer e and also the set E has some upper bound u. Let e' be any integer larger than u. Then working through the decreasing list of integers $e', e' - 1, e' - 2, \ldots, e + 1, e$, we come to a first one which is in E. This is clearly the maximum member of E. Similarly if E' is any non-empty set of integers which is bounded below then there exists an integer e in E' and another integer e' which is a lower bound of E'. Then working through the list of integers $e', e' + 1, e' + 2, \ldots, e - 1, e$, it is clear that the first one which is in E' is the minimum member of E'.

In particular any non-empty subset of $\mathbb{N}$ is bounded below (by 1 for example) and it follows that such a set will have a minimum member. □

The last part of that result is the cornerstone of 'the principle of mathematical induction', but before deducing that principle as a 'corollary' (or consequence) of the theorem we illustrate it by means of an example.

Example Let the numbers $x_1, x_2, x_3, \ldots$ be defined by

$x_1 = 2$ and $x_n = \sqrt{(6 + x_{n-1})}$ for $n > 1$.

Show that each of the numbers satisfies $2 \leqslant x_n < 3$.

Solution Consider the required property '$2 \leqslant x_n < 3$' as a property **of *n***. So $n = 1$ has the property since $2 \leqslant x_1 < 3$.

Hence

$$2 \leqslant x_1 < 3$$

But then

$$2 \leqslant \sqrt{(6 + 2)} \leqslant \underbrace{\sqrt{(6 + x_1)}}_{= x_2} < \sqrt{(6 + 3)} = 3$$

i.e.

$$2 \leqslant x_2 < 3$$

and so $n = 2$ has the property. But then

$$2 \leqslant \sqrt{(6 + 2)} \leqslant \underbrace{\sqrt{(6 + x_2)}}_{= x_3} < \sqrt{(6 + 3)} = 3$$

i.e.

$$2 \leqslant x_3 < 3$$

and so $n = 3$ has the property. But then

$$\vdots$$

i.e.

$$2 \leqslant x_{k-1} < 3$$

and so $n = k - 1$ has the property. But then

$$2 \leqslant \sqrt{(6+2)} \leqslant \underbrace{\sqrt{(6 + x_{k-1})}}_{= x_k} < \sqrt{(6+3)} = 3$$

i.e.

$$2 \leqslant x_k < 3$$

$$\vdots$$

and so the result holds for all positive integers. □

Once we have established the general 'principle of induction' the examples will be rather easier.

Corollary (*The principle of mathematical induction*) Let P be a property which may or may not hold for any positive integer n, and assume that:

(I) the number 1 has the property P;
(II) if any positive integer $k - 1$ has the property then so does the integer k.

Then it follows that the property P holds for all positive integers.

Proof Let F be the set of positive integers for which the property fails (we hope to show that F is the empty set).

By (I) the property holds for the number 1 and so that number is not in F. We can now deduce from the given conditions (I) and (II) the following fact about F:

If k is in F then $k - 1$ is also in F

(For if k is in F then $k > 1$ and by (II) we cannot have the property P holding for $k - 1$ but not for k.) Hence the set F has no minimum member. By the theorem the only subset of $\mathbb{N}$ which does not have a minimum member is the empty set. It follows that F is the empty set and that the property P holds for all positive integers. □

Example Show that for each positive integer n and for each real number $x \neq 1$

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

Solution Although there is an alternative method we shall use induction to establish the sum of this 'geometric progression' (or 'geometric series').

(I) In the case $n = 1$ the result is trivial, both sides of the required equation being 1.
(II) If the result is known for some positive integer $k - 1$ then

$$1 + x + x^2 + \cdots + x^{k-2} = \frac{1 - x^{k-1}}{1 - x}$$

and so

$$\begin{aligned} &1 + x + x^2 + \cdots + x^{k-1} \\ &= (1 + x + x^2 + \cdots + x^{k-2}) + x^{k-1} \\ &= \frac{1 - x^{k-1}}{1 - x} + x^{k-1} = \frac{1 - x^{k-1} + x^{k-1} - x^k}{1 - x} = \frac{1 - x^k}{1 - x} \end{aligned}$$

which is the required result in the case when $n = k$. Hence if the positive integer $k - 1$ works then so does k.

By the principle of induction established above the result therefore holds for every positive integer n. □

Example Show that any finite set (i.e. one which contains a finite number of members) is bounded.

Solution We aim to show that the set

$$\{x_1, x_2, \ldots, x_n\}$$

is bounded. You might think that this is obvious because you can put the n numbers in order and then the lowest forms a lower bound of the set and the highest forms an upper bound of the set. But in case you feel that such an argument involving 'n' items is a little dodgy we'll prove this result by induction.

(I) In the case $n = 1$ the set is $\{x_1\}$ which is clearly bounded below (by x_1) and bounded above (by x_1 too!).
(II) So now assume that the result is known for the positive integer $k - 1$; i.e. any set of $k - 1$ numbers is bounded. Consider the case when $n = k$:

$$\{x_1, x_2, \ldots, x_k\} = \underbrace{\{x_1, x_2, \ldots, x_{k-1}\}}_{\substack{\text{bounded (by our assumption}\\\text{concerning the } k-1 \text{ case)}}} \cup \underbrace{\{x_k\}}_{\substack{\text{bounded}\\\text{(by (I))}}}$$

which is bounded since (as we saw in exercise 4 above) the union of two bounded sets is bounded.

That completes the proof by induction and shows that a finite set $\{x_1, x_2, \ldots, x_n\}$ is bounded. □

Exercises

1 Which of the following sets has a maximum member and which has a minimum member?

$$\varnothing = \text{the empty set}$$
$$A = \{x \in \mathbb{Q}: 1 \leqslant x \leqslant \sqrt{2}\}$$
$$B = \{1/p: p \text{ is a prime})$$
$$C = \{1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{4}, 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}, 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}, \ldots\}$$
$$D = \{1, 1 - \tfrac{1}{3}, 1 - \tfrac{1}{3} - \tfrac{1}{9}, 1 - \tfrac{1}{3} - \tfrac{1}{9} - \tfrac{1}{27}, 1 - \tfrac{1}{3} - \tfrac{1}{9} - \tfrac{1}{27} - \tfrac{1}{81}, \ldots\}$$

2 Show that if a set has a maximum member then it has a supremum which is contained in the set. Conversely show that if the set has a supremum which is contained in it then the set has a maximum member. (So that a set has a maximum member **if and only if** it has a supremum contained in the set.)

State and prove a result connecting the infimum and the minimum of a set.

3 Let x and y be real numbers with $y > x + 1$. By considering the largest member of the set

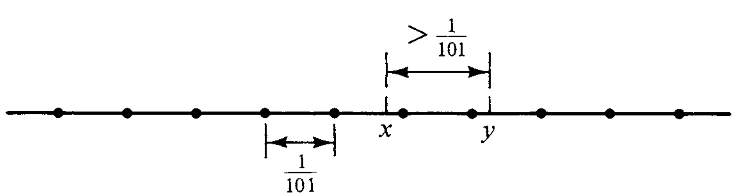
$$\{n \in \mathbb{Z}: n < y\}$$

show that there exists an integer between x and y.

4 We shall now try to establish that between any two different numbers there is a rational number. To try to see first an informal verification of this fact imagine for a moment that the two numbers are more than 1/101 apart. Then surely at least one of the fractions

$$\ldots \tfrac{-3}{101}, \tfrac{-2}{101}, \tfrac{-1}{101}, 0, \tfrac{1}{101}, \tfrac{2}{101}, \tfrac{3}{101}, \ldots$$

illustrated below would lie between the two numbers?



Formally suppose that x and y are two real numbers with $x < y$. Let N

be an integer chosen with $N > 1/(y - x)$. (How do we know that such an integer exists?) Use the previous exercise to show that there exists an integer M with $Nx < M < Ny$ and deduce that there is a rational number between x and y.

Deduce that between any two different numbers there is a rational and an irrational number.

5 Prove by induction that for each positive integer n

(i) $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{n(n+1)} = 1 - \dfrac{1}{n+1}$

(ii) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

6 Prove by induction that for each positive integer n

(i) $(1 + x)^n \geqslant 1 + nx$

where x is any number with $x \geqslant -1$. (Think carefully about where in your solution you need the fact that $x \geqslant -1$.)

This result is known as *Bernoulli's inequality* and is credited to Jacob Bernoulli, one of a great Swiss mathematical family of the seventeenth and eighteenth centuries.

(ii) $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$

where x is any number and $\binom{n}{r}$ denotes the 'binomial coefficient' and equals $n!/(r!(n-r)!)$. (You will need the fact that

$$\binom{k-1}{r-1} + \binom{k-1}{r} = \binom{k}{r}$$

which you can easily verify for yourselves.)

This result is known as the *binomial theorem.*

7 Prove by induction that for each positive integer n

(i) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \cdots + \dfrac{1}{2^{n-1}} \geqslant \dfrac{1}{2}(n+1)$

(ii) $1 + \dfrac{1}{2^r} + \dfrac{1}{3^r} + \dfrac{1}{4^r} + \dfrac{1}{5^r} + \cdots + \dfrac{1}{(2^n - 1)^r} \leqslant \dfrac{1 - (\frac{1}{2})^{(r-1)n}}{1 - (\frac{1}{2})^{r-1}}$ $(r \neq 1)$

(Solutions on page 227)

Some loose ends

Our final topics in this chapter concern two numbers about which you may be a little vague (if you're anything like I was when I started studying analysis) and yet which keep popping up in our studies: they are the numbers e and π. I used to wonder why 'logs to the base e' are called the *natural* logs when 10 seemed a much more natural base than e (≈ 2.71828). And why did we switch from the nice neat 180° in a straight line to the peculiar π (≈ 3.14159) 'radians'? These are questions which we shall answer in due course for it takes a fair bit of analysis before we can see these two numbers at work. But having struggled to set up the real numbers and to pin-point $\sqrt{2}$ it is fitting to end the chapter by establishing the existence of these two numbers.

Theorem Let E be the set of numbers

$$\left\{1, 1 + \frac{1}{1!}, 1 + \frac{1}{1!} + \frac{1}{2!}, 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}, 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}, \ldots\right\}$$

Then E is bounded above. Its least upper bound is called e and it is irrational.

Proof Note that

$$3! = 3 \times 2 > 2^2, \qquad 4! = 4 \times 3 \times 2 > 2^3, \qquad 5! = 5 \times 4 \times 3 \times 2 > 2^4 \text{ etc}$$

(and if you were very fussy you could establish the general result by using induction). Hence, by using the sum of a geometric progression (established in an earlier example) in the case $x = \frac{1}{2}$, we see that for $n > 2$

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3$$

Therefore every member of E is less than 3 and the set E is bounded above. It follows that E has a least upper bound, which we shall denote by e.

Before proceeding to prove that e is irrational note that for any integer $N > 2$

$$\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \frac{1}{(N+3)!} + \cdots + \frac{1}{(N+k)!}$$

$$= \frac{1}{(N+1)!}\left(1 + \frac{1}{(N+2)} + \frac{1}{(N+2)(N+3)} + \cdots \right.$$

$$\left. + \frac{1}{(N+2)(N+3)\ldots(N+k)}\right)$$

$$< \frac{1}{(N+1)!}\left(1 + \frac{1}{(N+2)} + \frac{1}{(N+2)^2} + \cdots + \frac{1}{(N+2)^{k-1}}\right)$$
$$< \frac{1}{(N+1)!}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right)$$
$$< \frac{2}{(N+1)!} = \frac{2}{N+1}\cdot\frac{1}{N!} \leqslant \frac{1}{2N!}$$

(the upper bound of 2 for the geometric progression following easily from the sum of a geometric progression in the earlier example in the case $x = \frac{1}{2}$).

We shall now show that e is irrational by assuming that it is rational and deducing a contradiction.

Assume then that e is the rational m/n where m and n are positive integers. Then by multiplying the top and bottom of this fraction by appropriate factors we can deduce that

$$e = \frac{M}{N!} \text{ for some integers } M \text{ and } N \text{ with } N > 2$$

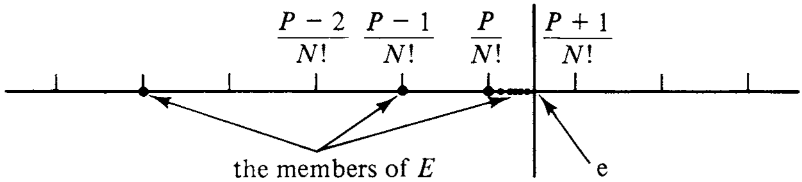
For example if e = 3 then e = 18/3! and if e = 14/5 then e = 336/5! Now

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} = \frac{P}{N!} \text{ say}$$

is a member of the set E. Also from the inequalities established above we can deduce that any larger member of the set E satisfies

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(N+k)!}$$
$$= \underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!}}_{= \frac{P}{N!}}$$
$$+ \underbrace{\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots + \frac{1}{(N+k)!}}_{< \frac{1}{2N!}} < \frac{P + \frac{1}{2}}{N!}$$

Let us digest those facts: $P/N!$ is a member of the set E and although there are bigger members than this none exceeds $(P + \frac{1}{2})/N!$. Let us illustrate the numbers of the form $1/N!$, $2N!$, $3/N!, \ldots$ together with members of the set E and the number e:



Thus e $= M/N!$, the least upper bound of the set E satisfies

$$\frac{P}{N!} < \frac{M}{N!} < \frac{P+1}{N!}$$

which is impossible since M is an integer. This contradiction shows that e is irrational as claimed. □

As mentioned earlier the number e will occur many times in our studies. The numbers in the set E defined above increase towards the supremum e, getting closer and closer to it. In exercise 1 below you are asked to use this fact to calculate an approximate value of e. The existence of π is derived in a similar way in the exercises.

Surely after all this work building the real numbers on firm foundations, we shall be ready to study functions in the next chapter?

Exercises

1 Use your calculator (or computer) to work out the approximate value of the first few members of the set E in the above theorem. Hence obtain an approximation for e.

2 Let P be the set of numbers

$$\{\sqrt{6}, \sqrt{(6(1+\tfrac{1}{4}))}, \sqrt{(6(1+\tfrac{1}{4}+\tfrac{1}{9}))}, \sqrt{(6(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}))},$$
$$\sqrt{(6(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}))}, \ldots\}$$

Use the fact that $1/m^2 < 1/m(m-1)$ together with exercise 5(i) on page 19 to show that the number $\sqrt{12}$ is an upper bound of P. The members of P are increasing towards its supremum: use your calculator or computer to work out some of the members of P and hence obtain an estimate of $\sup P$. (In fact the supremum is π, which arises in other much more natural ways as we shall see in later chapters. And we shall prove in the very last exercise of the book that π is irrational.)

3 In the two previous exercises we have used our calculators to obtain approximations to the supremum of a set, but only after we had proved mathematically that the supremum existed (by showing that each