1

Matroids and Rigid Structures

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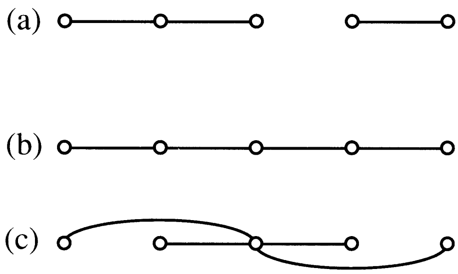
Many engineering problems lead to a system of linear equations – a represented matroid – whose rank controls critical qualitative features of the example (Sugihara, 1984; 1985; White & Whiteley, 1983). We will outline a selection of such matroids, drawn from recent work on the rigidity of spatial structures, reconstruction of polyhedral pictures, and related geometric problems.

For these situations, the combinatorial pattern of the example determines a sparse matrix pattern that has both a generic rank, for general 'independent' values of the non-zero entries, and a geometric rank, for special values for the coordinates of the points, lines, and planes of the corresponding geometric model. Increasingly, the generic rank of these examples has been studied by matroid theoretic techniques. These geometric models provide nice illustrations and applications of techniques such as matroid union, truncation, and semimodular functions. The basic unsolved problems in these examples highlight certain unsolved problems in matroid theory. Their study should also lead to new results in matroid theory.

1.1. Bar Frameworks on the Line – the Graphic Matroid

We begin with the simplest example, which will introduce the vocabulary and the basic pattern. We place a series of distinct points on a line, and specify certain *bars* – pairs of joints which are to maintain their distance – defining a *bar framework on the line*. We ask whether the entire framework is 'rigid' – i.e. does any motion of the joints along the line, preserving these distances, give all joints the same velocity, acceleration, etc.? Clearly a framework has an *underlying graph* $G=(V, E)$, with a vertex v_i for each joint p_i and an undirected edge $\{i, j\}$ for each bar $\{p_i, p_j\}$. In fact, we describe the framework as $G(\mathbf{p})$, where G is a graph without multiple edges or loops, and $\mathbf{p}$ is an assignment of points p_i to the vertices v_i. If this graph is not connected,

Figure 1.1.



then each component can move separately in the framework, and the framework is not rigid (Figure 1a). Conversely, a connected graph always leads to a rigid framework (Figure 1.1b), since each bar ensures that its two joints have the same motion on the line. This gives an informal proof of the following result.

1.1.1. Proposition. *A bar framework $G(\mathbf{p})$ on the line is rigid if and only if the underlying graph G is connected.*

To extract a matrix, we make this argument a little more formal. Assume the joints p_i move along smooth paths $p_i(t)$. The length of a bar $\|p_i(t)-p_j(t)\|$, and its square, remain constant. If we differentiate, this condition becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[p_i(t)-p_j(t)]^2=[p_i(t)-p_j(t)][p_i'(t)-p_j'(t)]=0.$$

At $t=0$, this is written $(p_i-p_j)(p_i'-p_j')=0$. If we have distinct joints on the line, so that $(p_i-p_j)\neq 0$, this simplifies to $(p_i'-p_j')=0$.

With this in mind, we define an *infinitesimal motion* of a bar framework on the line $G(\mathbf{p})$ as an assignment of a velocity u_i along the line to each joint p_i such that $u_i-u_j=0$ for each bar $\{v_i, v_j\}$. For example, consider the framework in Figure 1.1c. The four bars lead to four equations in the unknowns $\mathbf{u}=(u_1, u_2, u_3, u_4, u_5)$:

$$\begin{bmatrix}1 & 0 & -1 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}.$$

In general, this system of linear equations is written $R(G, \mathbf{p})\times\mathbf{u}^t=0$, where the *rigidity matrix* $R(G, \mathbf{p})$ has a row for each edge of the graph and a column for each vertex, and $\mathbf{u}^t$ is the transpose of the vector of velocities. We note that $R(G, \mathbf{p})$ is the transpose of the usual matrix representation for the graph over the reals: the rows are independent in $R(G, \mathbf{p})$ if and only if the corresponding edges are a forest (an independent set of edges in the cycle matroid of the graph).

A *trivial infinitesimal motion* is the derivative of a rigid motion of the line – i.e. a translation with all velocities equal. These form a one-dimensional subspace of the solutions. *An infinitesimally rigid framework on the line* has only these trivial infinitesimal motions, so the rigidity matrix has rank $|V| - 1$. This rank corresponds to a spanning tree on the vertices, or a basis for the cycle matroid of the complete graph on $|V|$ vertices. This proves the following infinitesimal version of Proposition 1.1.1.

1.1.2. Proposition. *A bar framework $G(\mathbf{p})$ on the line is infinitesmially rigid if and only if the underlying graph G is connected.*

1.2. Bar Frameworks in the Plane

A *bar framework in the plane* is a graph $G = (V, E)$ and an assignment $\mathbf{p}$ of points $\mathbf{p}_i \in \mathbb{R}^2$ to the vertices v_i such that $\mathbf{p}_i \neq \mathbf{p}_j$ if $\{i, j\} \in E$. If we differentiate the condition that bars have constant length in any smooth motion, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{p}_i(t) - \mathbf{p}_j(t)]^2 = [\mathbf{p}_i(t) - \mathbf{p}_j(t)]\cdot[\mathbf{p}_i'(t) - \mathbf{p}_j'(t)] = 0.$$

Accordingly, an *infinitesimal motion* of plane bar framework is an assignment $\mathbf{u}$ of velocities $\mathbf{u}_i \in \mathbb{R}^2$ to the joint such that

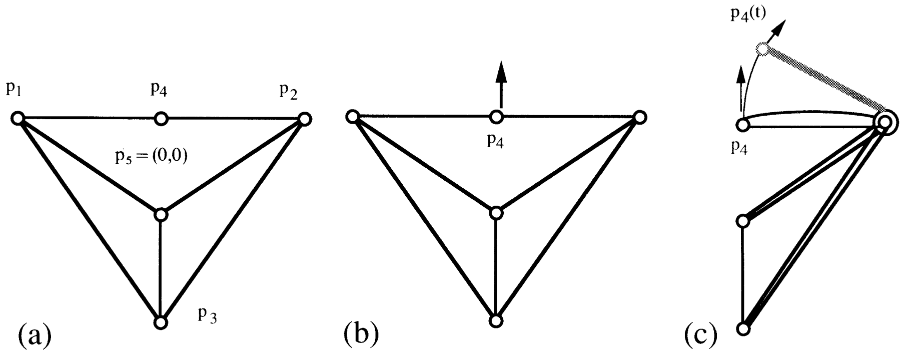
$$(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j) = 0 \quad \text{for each } \{i, j\} \in E.$$

A plane bar framework is *infinitesimally rigid* if all infinitesimal motions are *trivial*: $\mathbf{u}_i = \mathbf{s} + \beta(\mathbf{p}_i)^{\perp}$, where $\mathbf{s}$ is a fixed translation vector, $(\mathbf{x}, \mathbf{y})^{\perp} = (\mathbf{y}, -\mathbf{x})$ rotates the vector 90° counterclockwise, and $\beta(\mathbf{p}_i)^{\perp}$ represents a rotation about the origin. (These infinitesimal rotations and translations are the derivatives of smooth rigid motions of the plane.)

The system of equations for an infinitesimal motion has the form $R(G, \mathbf{p}) \times \mathbf{u}^t = 0$, where the *rigidity matrix* $R(G, \mathbf{p})$ now has a row for each edge of the graph and two columns for each vertex. The row for edge $\{i, j\}$ has the form

$$[0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad \mathbf{p}_i - \mathbf{p}_j \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad \mathbf{p}_j - \mathbf{p}_i \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0]$$

Figure 1.2.



1.2.1. Example. Consider the frameworks in Figure 1.2. The framework of Figure 1.2a gives the rigidity matrix

$$
\begin{array}{c}
\{1,3\}\\ \{1,4\}\\ \{1,5\}\\ \{2,3\}\\ \{2,4\}\\ \{2,5\}\\ \{3,5\}
\end{array}
\begin{bmatrix}
x_1-x_3 & y_1-y_3 & 0 & 0 & x_3-x_1 & y_3-y_1 & 0 & 0 & 0 & 0\\
\tfrac{1}{2}(x_1-x_2) & \tfrac{1}{2}(y_1-y_2) & 0 & 0 & 0 & 0 & \tfrac{1}{2}(x_2-x_1) & \tfrac{1}{2}(y_2-y_1) & 0 & 0\\
x_1 & y_1 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -y_1\\
0 & 0 & x_2-x_3 & y_2-y_3 & x_3-x_2 & x_3-y_2 & 0 & 0 & 0 & 0\\
0 & 0 & \tfrac{1}{2}(x_2-x_1) & \tfrac{1}{2}(y_2-y_1) & 0 & 0 & \tfrac{1}{2}(x_1-x2) & \tfrac{1}{2}(y_1-y2) & 0 & 0\\
0 & 0 & x_2 & y_2 & 0 & 0 & 0 & 0 & -x_2 & -y_2\\
0 & 0 & 0 & 0 & x_3 & y_3 & 0 & 0 & -x_3 & -y_3
\end{bmatrix}
$$

The rows of this matrix are dependent and have rank 6. This leaves a $(10-6=4)$-dimensional space of infinitesimal motions, including the non-trivial motion shown in Figure 1.2b, which assigns zero velocity to all joints but $\mathbf{p}_4$, and gives $\mathbf{p}_4$ a velocity perpendicular to the bars at $\mathbf{p}_4$. Thus the framework is not infinitesimally rigid.

The infinitesimal motion is not the derivative of some smooth path for the vertices. The framework is *rigid* – all smooth paths, or even continuous paths, give frameworks congruent to the original framework. Figure 1.2c gives a similar framework which has the same infinitesimal motions, but is not rigid.

These examples show that there is a difference in the plane between rigid frameworks and infinitesimally rigid frameworks. A *non-rigid* plane framework will have an analytic path of positions $\mathbf{p}(t)=(\ldots,\mathbf{p}_i(t),\ldots)$, with all bar lengths of $\mathbf{p}(t)$ the same as bars in $\mathbf{p}(0)$, but $\mathbf{p}(t)$ not congruent to $\mathbf{p}(0)$, for all $0<t<1$ (Figure 1.2c). The first non-zero derivative of this path will be a non-trivial infinitesemal motion. However, the converse is false: many infinitesimal motions are not the derivative of an analytic path (recall Figure 1.2b). For any framework, the independence of the rows of the rigidity matrix induces a matroid on the edges of the graph. If ‘rigidity’ in a particular plane framework were used to define an independence structure on the edges of a graph, this need not be a matroid (see Exercise 1.6). Therefore, we will restrict ourselves, throughout this chapter, to the simpler concepts of infinitesimal motions and infinitesimal rigidity.

The space of trivial plane infinitesimal motions has dimension 3, for frameworks with at least two distinct joints. This space can be generated by two translations in distinct directions and a rotation about any fixed point. Thus an infinitesimally rigid framework with more than two joints will have an $|E|$ by $2|V|$ rigidity matrix of rank $2|V|-3$. Our basic problem is to determine which graphs G allow this matrix to have rank $2|V|-3$ for at least some plane frameworks $G(\mathbf{p})$.

The independence structure of the rows of the rigidity matrix defines a matroid on the edges of the complete graph on the vertices. This matroid depends on the positions of the joints. If we vary the positions there are 'generic' positions that give a maximal collection of independent sets (for example, positions where the coordinates are algebraically independent real numbers). At these positions we have the *generic rigidity matroid for* $|V|$ *vertices in the plane.*

1.2.2. Example. Consider the framework in Figure 1.3a. With vertices as indicated we have the rigidity matrix

$$\begin{array}{r} (a_1, b_1) \\ (a_1, b_2) \\ (a_1, b_3) \\ (a_2, b_1) \\ (a_2, b_2) \\ (a_2, b_3) \\ (a_2, b_1) \\ (a_3, b_2) \\ (a_3, b_3) \end{array}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}$$

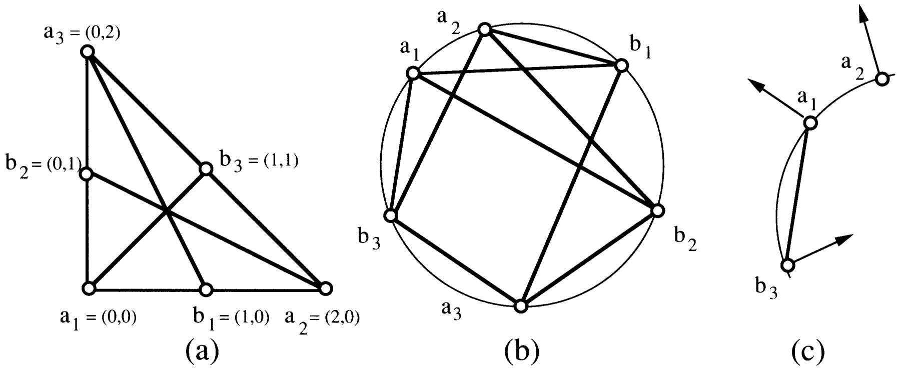
The graph of the framework has $|E| = 2|V| - 3$, so the framework is infinitesimally rigid if and only if the rows are independent. This independence can be checked by deleting the final three columns and seeing that the determinant of the 9×9 submatrix is non-zero. This framework is infinitesimally rigid and the graph is generically rigid, and generically independent.

Consider any realization with distinct joints $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ on a unit circle centred at the origin (Figure 1.3b). This has a non-trivial 'in–out' infinitesimal motion (Figure 1.3c):

$$\text{for joints } \mathbf{a}_i \text{ take the velocity } \mathbf{a}_i' = \mathbf{a}_i;$$

$$\text{for joints } \mathbf{b}_j \text{ take the velocity } \mathbf{b}_j' = -\mathbf{b}_j.$$

Figure 1.3.



These velocities preserve the length of all bars $(\mathbf{a}_i, \mathbf{b}_j)$, since

$$(\mathbf{a}_i - \mathbf{b}_j) \cdot (\mathbf{a}'_i - \mathbf{b}'_j) = (\mathbf{a}_i - \mathbf{b}_j) \cdot (\mathbf{a}_i + \mathbf{b}_j) = (\mathbf{a}_i) \cdot (\mathbf{a}_i) - (\mathbf{b}_j) \cdot (\mathbf{b}_j) = 1 - 1 = 0.$$

This infinitesimal motion is non-trivial. Letting $\theta \neq 0$ be the angle between the unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we show that the distance $(\mathbf{a}_1 - \mathbf{a}_2)$ is changing instantaneously:

$$(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{a}'_1 - \mathbf{a}'_2) = (\mathbf{a}_1) \cdot (\mathbf{a}_1) - 2(\mathbf{a}_1) \cdot (\mathbf{a}_2) + (\mathbf{a}_2) \cdot (\mathbf{a}_2) = 1 + 1 - 2\cos\theta > 0.$$

Thus this special position is not generic (see Exercise 1.9).

We want to characterize the graphs of *isostatic plane frameworks* – minimal infinitesimally rigid frameworks in the sense that removing any one bar introduces a non-trivial infinitesimal motion. These graphs, of size $|E| = 2|V| - 3$, are the bases of the generic rigidity matroid 'of the complete graph' on the set of vertices.

Thus an isostatic framework corresponds to a row basis for the rigidity matrix of any infinitesimally rigid framework extending the framework. The independence of such a set of edges is determined by maximal minors of the rigidity matrix. This independence is *generic* in the sense that these minors are non-zero polynomials in the positions $\mathbf{p}_i$. If such a polynomial is non-zero for some position $G(\mathbf{p})$, then almost all $\mathbf{q} \in \mathbb{R}^{2|V|}$ give isostatic frameworks $G(\mathbf{q})$ (see Section 1.7).

More surprisingly, for points where this matrix and all its minors have the maximal rank achieved for $\mathbf{q} \in \mathbb{R}^{2|V|}$, infinitesimal rigidity and any reasonable form of local rigidity actually coincide (see, for example, Exercise 1.7).

We note that throughout this chapter the generic matroids defined on the complete graph of $|V|$ vertices are symmetric on the vertices – any permutation of the vertices does not change the independence of a set of edges. As a convention, we write the attached vertices for a subset of edges E' as V'.

1.2.3. Theorem. *For a graph G, with at least two vertices, the following are equivalent conditions:*

(i) *G has some positions $G(\mathbf{p})$ as an isostatic plane framework;*

(ii) $|E| = 2|V| - 3$ *and for all proper subsets of edges E' incident with vertices* V', $|E'| \leqslant 2|V'| - 3$;

(iii) *adding any edge to E (including doubling an edge) gives an edge set covered by two edge-disjoint spanning trees.*

Proof. (i)⇒(ii): For an isostatic plane framework $G(\mathbf{p})$ on at least two vertices, the rows of the rigidity matrix have rank $|E| = 2|V| - 3$. If any proper subset of edges has $|E'| > 2|V'| - 3$, the corresponding rows are dependent. Since $G(\mathbf{p})$ is independent, we conclude that $|E'| \leqslant 2|V'| - 3$ for all proper subsets.

(ii)⇔(iii): The count $f(E') = 2|V'| - 3$ defines a non-decreasing semimodular function on sets of edges, which is non-negative on non-empty sets (see Exercise 1.1). This semimodular function defines a matroid by the standard property:

$$E \text{ is independent if and only if } |E'| \leqslant f(E') \text{ for all proper subsets } E'. \quad (1.1)$$

This count has the form $f(E') = (2|V'| - 2) - 1$ which shows that the matroid for f is a Dilworth truncation of the matroid defined by the semimodular function $g(E') = 2(|V'| - 1)$. In turn, the semimodular function g represents a matroid union of two copies of the matroid given by the semimodular function $h(E') = |V| - 1$ (the cycle matroid of the graph). Thus a graph is independent in the matroid of f if and only if adding any edge (including doubling an edge) gives a graph covered by two edge-disjoint forests.

Before we prove (iii)⇒(i), we need a lemma about a simpler matrix that has rank $2|V| - 2$ (matching the function g). For a graph $G = (V, E)$, including possible multiple edges, a *2-frame* $G(\mathbf{d})$ is an assignment of directions $\mathbf{d}_e \in \mathbb{R}^2$ to the edges. An *infinitesimal motion of the 2-frame* $G(\mathbf{d})$ is an assignment of velocities $\mathbf{u}_i \in \mathbb{R}^2$ to the vertices such that

$$\mathbf{d}_e \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0 \quad \text{for every edge } e \text{ joining } v_i \text{ and } v_j \ (i < j).$$

This system of equations defines the *rigidity matrix* $R(G, \mathbf{d})$ for the 2-frame.

1.2.4. Lemma. *The rows of the rigidity matrix of a generic 2-frame $G(\mathbf{d})$ are independent if and only if G is the union of two edge-disjoint forests.*

Proof. Take the two forests F_1 and F_2. For all edges in the first forest, we assign the direction (1, 0). For all edges in the second forest, we assign the direction (0, 1). If we reorder the rows and columns of this rigidity matrix, placing all second columns of vertices to the right, and all rows for the second forest at the bottom, we have a pattern:

$$\begin{bmatrix} [F_1] & [0] \\ [0] & [F_2] \end{bmatrix}$$

where $[F_1]$ and $[F_2]$ are the standard matrices representing the two forests over the reals. Thus the blocks $[F_1]$ and $[F_2]$ have non-zero minors on all their rows, and the entire matrix also has a non-zero minor on all the rows. We conclude that the rows are independent.

Conversely, if the rows are independent, we can reorder the columns as above. The independence guarantees a non-zero minor on all the rows. Using a Laplace expansion on the two blocks, there are non-zero minors on complementary sets of rows. Each of these non-zero minors on the matrix representing the graphic matroid must correspond to a forest, as required. □

Proof of Theorem 1.2.3 (*continued*). (iii)$\Rightarrow$(i) Assume that adding any edge to E (including doubling an edge) gives an edge set covered by two edge-disjoint forests. We show that this gives independent rows in the rigidity matrix for some (almost all) choices of the points.

Take a 2-frame $G(\mathbf{d})$ with algebraically independent directions for the edges. By our assumption, adding any edge (or doubling any edge) between any pair of vertices gives an independent 2-frame E^*. Therefore, the 2-frame on E has an infinitesimal motion $\mathbf{u}_{ij}$ that has different velocities on the two vertices of the added edge. Taking linear combinations of these $\mathbf{u}_{ij}$ there is an infinitisemal motion $\mathbf{u}$ that assigns distinct velocities $\mathbf{u}_i = (s_i, t_i)$ to each of the vertices of G.

To create the independent framework $G(\mathbf{p})$, we set $\mathbf{p}_i = (-t_i, s_i)$. Since $\mathbf{u}_i \neq \mathbf{u}_j$, we have $\mathbf{p}_i \neq \mathbf{p}_j$ for each edge $\{i, j\}$, as required in a framework. We claim that the rigidity matrix of $G(\mathbf{p})$ has rows parallel to the rows of the original 2-frame $G(\mathbf{d})$. Clearly

$$(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j) = (-t_i + t_j,\ s_i - s_j)\cdot(s_i - s_j,\ t_i - t_j) = 0 = \mathbf{d}_e\cdot(\mathbf{u}_i - \mathbf{u}_j).$$

Since $\mathbf{u}_i - \mathbf{u}_j \neq \mathbf{0}$, this means $(\mathbf{p}_i - \mathbf{p}_j) = \beta_e \mathbf{d}_e$ for some non-zero scalar β_e. The rigidity matrix of the framework is equivalent to the rigidity matrix of the independent 2-frame. We conclude that a set of edges satisfying condition (1.1) has been realized as an independent (therefore isostatic) bar framework. (The infinitesimal motion $\mathbf{u}$ of the 2-frame is a rotation of this framework around the origin.)

This completes the proof. □

Figure 1.4 shows some other examples of the graphs of isostatic frameworks in the plane (Figure 1.4a) and graphs of circuits in the plane generic rigidity matroid (Figure 1.4b) with $|E| = 2|V| - 2$, and $|E'| \leqslant 2|V'| - 3$ for proper subsets.

The semimodular count of Theorem 1.2.3 (ii) converts to a criterion for graphs of infinitesimally rigid frameworks. We state the theorem without proof.

1.2.5. Corollary. (*Lovász & Yemini, 1982*) *A graph has realizations as an infinitesimally rigid plane framework if and only if for every partition of the* *edges into non-empty subsets* $(\ldots E^j \ldots)$, *with vertices* V^j *incident with the edges* E^j,

$$\sum_j (2|V^j| - 3) \geqslant 2|V| - 3.$$

Figure 1.5 gives a simple example of a 5-connected graph, in a vertex sense, which is never infinitesimally rigid by this criterion: take the eight K_5 graphs as sets of the partition, and all other edges as singletons and apply Corollary 1.2.5. However, every graph which is 6-connected in a vertex sense is generically rigid (Exercise 1.16).

The dependence of rows in the rigidity matrix, or *dependence of bars in the framework*, also has a physical interpretation. A *self-stress* on a framework is an assignment of scalars ω_{ij} to the bars $\{\mathbf{p}_i, \mathbf{p}_j\}$ such that for each joint $\mathbf{p}_i$ there is an equilibrium (Figures 1.6a, b):

Figure 1.4. (a) $|E| = 2|V| - 3$, $|E'| \leqslant 2|V'| - 3$; (b) $|E| = 2|V| - 2$, $|E'| \leqslant 2|V'| - 3$.

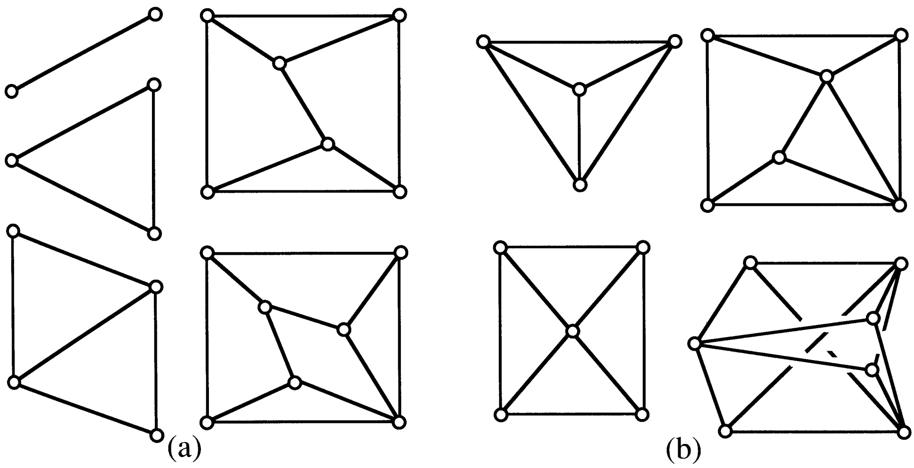


Figure 1.5.

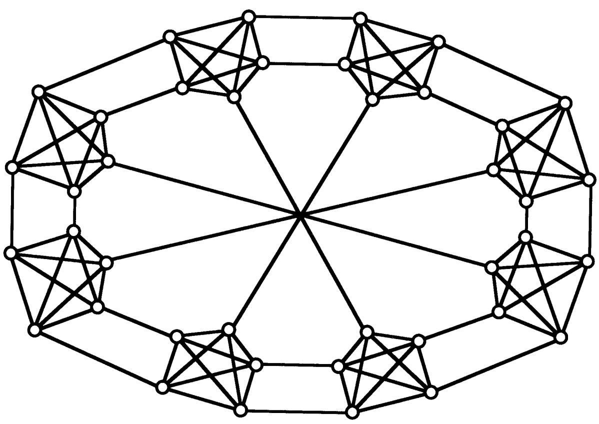
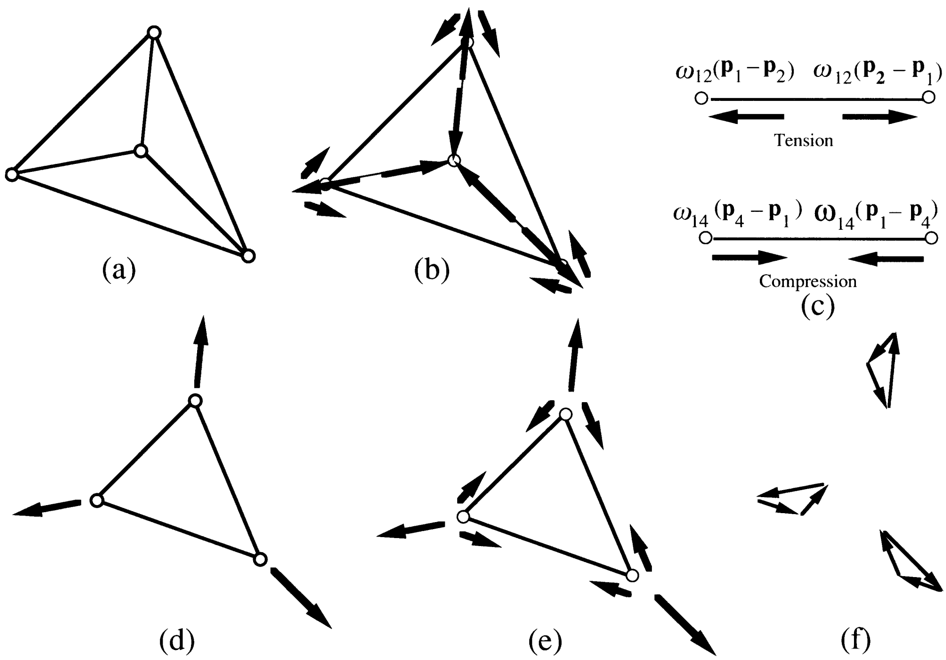


Figure 1.6.



$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = \mathbf{0} \quad \text{(sum over all edges } \{i, j\} \text{ incident with } \mathbf{p}_i\text{).}$$

Thus a self-stress is equivalent to a row dependence. If $\omega_{ij} < 0$, we interpret this as a *compression* in the bar – a force $\omega_{ij}(\mathbf{p}_j - \mathbf{p}_i)$ at $\mathbf{p}_i$, and a force $\omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)$ at $\mathbf{p}_j$. If $\omega_{ij} > 0$, this is a *tension* in the bar (Figure 1.6c).

In the same spirit, the row space of the rigidity matrix is interpreted as the space of loads $\mathbf{L}_i$ *resolved* by forces in the bars of the framework (Figure 1.6d, e, f):

$$\mathbf{L}_i + \sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = \mathbf{0} \quad \text{(sum over all edges } \{i, j\} \text{ incident with } \mathbf{p}_i\text{).}$$

These resolved loads satisfy an additional property of global static equilibrium, defined below (see Exercise 1.17). Thus an *equilibrium load* is an assignment $\mathbf{L}_i$ of vectors to the vertices that satisfies the three equilibrium equations

$$\sum_i \mathbf{L}_i = \mathbf{0} \quad \text{and} \quad \sum_i \mathbf{L}_i \times \mathbf{p}_i = \mathbf{0}$$

where $\times$ represents a cross product in 3-space. A framework is *statically rigid* if all equilibrium loads on its joints are resolved.

We note that a single point is trivially both infinitesimally rigid and statically rigid in the plane. A single bar has only a one-dimensional space of equilibrium loads: $\alpha(\mathbf{p}_1 - \mathbf{p}_2)$ at $\mathbf{p}_1$ and $\alpha(\mathbf{p}_2 - \mathbf{p}_1)$ at $\mathbf{p}_2$. Since these are