Chapter 0. INTRODUCTORY DISCUSSIONS.

In the present introductory chapter we give comprehensive discussions of a variety of nonrelated topics. All of these bear on the concept of pseudo-differential operator, at least in the author's mind. Some are only there to make studying ψdo's appear a natural thing, reflecting the author's inhibitions to think along these lines.

In sec.1 we discuss the elementary facts of the Fourier transform, in sec.'s 2 and 3 we develop Fourier-Laplace transforms of temperate and nontemperate distributions. In sec.4 we discuss the Fourier-Laplace method of solving initial-value problems and free space problems of constant coefficient partial differential equations. Sec.5 discusses another problem in PDE, showing how the solving of an abstract operator equation together with results on hypo-ellipticity and "boundary-hypo-ellipticity" can lead to existence proofs for classical solutions of initial-boundary problems. Sec.6 is concerned with the operator e^{Lt}, for a first order differential expression L . Sec.'s 7 and 8 deal with the concept of characteristics of a linear differential expression and learning how to solve a first order PDE. Sec.9 gives a mini-introduction to Lie groups, focusing on the mutual relationship between Lie groups and Lie algebras. (Note the relation to ψdo's discussed in ch.8).

We should expect the reader to glance over ch.0 and use it to have certain prerequisites handy, or to get oriented in the serious reading of later chapters.

0. Some special notations.

The following notations, abbreviations, and conventions will be used throughout this book.

(a) $\kappa_n = (2\pi)^{-n/2}$, $đx = \kappa_n dx_1 dx_2 \ldots dx_n = \kappa_n dx$.

(b) $\langle x \rangle = (1+|x|^2)^{1/2}$, $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$, etc.

(c) Derivatives are written in various ways, at convenience: For $u=u(x)=u(x_1,\dots,x_n)$ we write $u^{(\alpha)}=\partial_x^\alpha u=\partial_{x^1}^{\alpha^1}\partial_{x^2}^{\alpha^2}\dots u =$ $=\partial^{\alpha_1}/\partial x^{\alpha_1}\dots\partial^{\alpha_n}/\partial x^{\alpha_n}u$. Or, $u_{|x_j}=\partial_{x_j}u$, $u_{|x}$ to denote the n-vector with components $u_{|x_j}$, $\nabla_x^k u$ for the k-dimensional array with components $u_{|x_{i_1}x_{i_2}\dots}$. For a function of $(x,\xi)=(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ it is often convenient to write $u_{(\beta)}^{(\alpha)}=\partial_\xi^\alpha\partial_x^\beta u$.

(d) A multi-index is an n-tuple of integers $\alpha=(\alpha_1,\dots,\alpha_n)$. We write $|\alpha|=|\alpha_1|+\dots+|\alpha_n|$, $\alpha!=\alpha_1!\dots\alpha_n!$, $\binom{\alpha}{\beta}=\binom{\alpha}{\beta}\dots\binom{\alpha}{\beta}$, $x^\alpha= x_1^{\alpha_1}\dots x_n^{\alpha_n}$, etc., $\mathbb{N}^n$={all multi-indices} .

(e) Some standard spaces: $\mathbb{R}^n$ = n-dimensional Euclidean space $\mathbb{B}^n$=directional compactification of $\mathbb{R}^n$ (one infinite point ∞x added in every direction (of a unit vector x).

(f) Spaces of continuous or differentiable complex-valued functions over a domain or differentiable manifold X (or sometimes only $X=\mathbb{R}^n$): C(X) = continuous functions on X ; CB(X)= bounded continuous functions on X; CO(X)= continuous functions on X vanishing at ∞ ; CS(X) = continuous functions with directional limits; $C_0(X)$ = continuous functions with compact support; $C^k(X)$= functions with derivatives in C, to order k, (incl. $k=\infty$). $CB^\infty(X)$="all derivatives exist and are bounded". The Laurent-Schwartz notations $\mathcal{D}(X)=C_0^\infty(X)$, $\mathcal{E}(X)=C^\infty(X)$ are used. Also $\mathcal{S}=\mathcal{S}(\mathbb{R}^n)$= "rapidly decreasing functions" (All derivatives decay stronger as any power of x). Also, distribution spaces $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}'$.

(g) L^p-spaces: For a measure space X with measure $d\mu$ we write $L^p(X)=L^p(X,d\mu)$={measurable functions u(x) with $|u|^p$ integrable} for $1\le p<\infty$; $L^\infty(X)$={essentially bounded functions}.

(h) Maps between general spaces: C(X,Y) denotes the continuous maps $X\to Y$. Similar for the other symbols under (f), i.e., CB(X,Y) ,.... .

(i) Classes of linear operators (X= Banach space) : $L(X)$ ($K(X)$)= continuous (compact) operators; $GL(X)$ ($U(H)$) = invertible (unitary) operators of $L(X)$ (of $L(H)$, H=Hilbert space); $U_n=U(\mathbb{C}_n)$. For operators $X\to Y$, again, $L(X,Y)$, etc.

(j) The convolution product: For $u,v\in L^1(\mathbb{R}^n)$ we write $w(x)$ $=(u*v)(x)=\kappa_n\int dy\,u(x-y)v(y)$ (Note the factor $\kappa_n=(2\pi)^{-n/2}$).

(k) Special notation: " $X\subset\subset Y$ " means that X is contained in a compact subset of Y .

(l) For technical reason we may write $\lim_{\varepsilon\to 0} a(\varepsilon) = a|_{\varepsilon\to 0}$.

(m) Abbreviations used: ODE (PDE) = ordinary (partial) differential equation (or "expression"). FOLPDE (or folpde)= first order linear partial differential equation (or "expression"); ψdo= pseudodifferential operator.

(n) Integrals need not be existing (proper or improper) Riemann or Lebesgue integrals, unless explicitly stated, but may be distribution integrals By this term we mean that either (i) the integral may be interpreted as value of a distribution at a testing function-the integrand may be a distribution, or (ii) the limit of Riemann sums exists in the sense of weak convergence of a sequence of (temperate) distributions, or (iii) the limit defining an improper Riemann integral exists in the sense of weak convergence, as above, or (iv) the integral may be a 'finite part' (cf. I,4).

(o) Adjoints: For a linear operator A we use 'distribution adjoint' $A^{\sim}$ and 'Hilbert space adjoint' A^*, corresponding to transpose A^T and adjoint $\bar{A}^T=A^*$, in case of a matrix $A=((a_{jk}))$, respectively. For a symbols $a(x,\xi)$, a^* (or a^+) may denote the symbol of the adjoint ψdo of $a(x,D)$, as specified in each section.

(p) supp u (sing supp u (or s.s.u)) denotes the (singular) support of the distribution u.

1. The Fourier transform; elementary facts.

Let $u \in L^1(\mathbb{R}^n)$ be a complex-valued integrable function. Then we define the Fourier transform $u^\wedge = Fu$ of u by the integral

$$u^\wedge(x) = \int_{\mathbb{R}^n} d\!\!\bar{}\,\xi\, u(\xi) e^{-ix\xi} \quad , \ x \in \mathbb{R}^n , \tag{1.1}$$

with $x\xi = x.\xi = \sum_{j=1}^n x_j \xi_j$, an existing Lebesgue integral. Clearly,

$$|u^\wedge(x)| \le \|u\|_{L_1} = \int_{\mathbb{R}^n} d\!\!\bar{}\,x\, |u(x)| \ . \tag{1.2}$$

Note that $u^\wedge$ is uniformly continuous over $\mathbb{R}^n$: We get

$$\begin{aligned} |u^\wedge(x)-u^\wedge(y)| &\le 2\int d\!\!\bar{}\,\xi\, |\sin(x-y)\xi/2|\,|u(\xi)| \\ &\le N|x-y|\,\|u\|_{L^1} + 2\int_{|\xi|\ge N} d\!\!\bar{}\,\xi\, |u(\xi)| \ , \end{aligned} \tag{1.3}$$

where the right hand side is $<\varepsilon$ if N is chosen for $\int_{|\xi|\ge N} < \varepsilon/4$, and then $|x-y| < \varepsilon/(2N\|u\|_{L^1})$. Moreover, we get $u^\wedge \in CO(\mathbb{R}^n)$, i.e., $\lim_{|x|\to\infty} u^\wedge(x)=0$, a fact, known as the Riemann-Lebesgue lemma.

To prove the latter, we reduce it to the case of $u \in C_0^\infty(\mathbb{R}^n)$: The space C_0^∞ is known to be dense in L^1. By (1.1) we get

$$(1.4) \qquad |u^\wedge(x)-v^\wedge(x)| \le \|u-v\|_{L^1} < \varepsilon/2 \ , \text{ as } v \in C_0^\infty \ , \ \|u-v\|_{L^1}<\varepsilon/2.$$

Hence $\lim_{|x|\to\infty} v^\wedge(x)=0$ implies $|u^\wedge| \le |u^\wedge - v^\wedge| + |v^\wedge| < \varepsilon$ whenever x is chosen according to $|v^\wedge| < \varepsilon/2$.

But for $v \in C_0^\infty$ the Fourier integral extends over a ball $|\xi| \le N$ only, since $v=0$ outside. We may integrate by parts for

$$(1.5) \qquad |x|^2 u^\wedge(x) = -\int \Delta_\xi (e^{-ix\xi}) v(\xi) d\!\!\!/\xi = -\int d\!\!\!/\xi e^{-ix\xi} (\Delta v)(\xi) = -(\Delta v)^\wedge(x) \ ,$$

with the Laplace differential operator $\Delta_\xi = \sum_{j=1}^n \partial_{\xi_j}^{\,2}$. Clearly we have $\Delta v \in C_0^\infty \subset L^1$ as well, whence (1.1) applies to Δv, for

$$(1.6) \qquad |v^\wedge(x)| \le \|\Delta v\|_{L^1}/|x|^2 \to 0 \ , \text{ as } |x| \to \infty \ ,$$

completing the proof.

The above partial integration describes a general method to be applied frequently in the sequel. (1.6) may be derived under the weaker assumptions that $v\in C^2$, and that all derivatives $v^{(\alpha)}$, $|\alpha|\le 2$, are in L^1 (cf. pbm.5). On the other hand, there are simple examples of $u\notin L^1$ such that $u^\wedge$ does not decay as rapidly as (1.6) indicates. In particular, $u\in L^1$ exists with $u^\wedge \notin L^1$ (cf.pbm.4).

This matter becomes important if we think of inverting the linear operator $F:L^1 \to CO$ defined by (1.1), because formally an inverse seems to be given by almost the same integral. Indeed, define the (complex) conjugate Fourier transform $\bar{F}:L^1 \to CO$ by $\bar{F}u = \overline{(Fu)}$, or, $u^\vee = \bar{F}u$, where

$$(1.7) \qquad u^\vee(x) = \int d\!\!\!/\xi e^{ix\xi} u(\xi) \ , \ u \in L^1(\mathbb{R}^n) \ .$$

Then, in essence, it will be seen that $\bar{F}$ is the inverse of the operator F. More precisely we will have to restrict F to a (dense) subspace of L^1, for this result. Or else, the definition of the operator $\bar{F}$ must be extended to certain non-integrable functions, for which existence of the Lebesgue integral (1.7) cannot be expected. Both things will be done, eventually.

It turns out that F induces a unitary operator of the Hilbert space $L^2(\mathbb{R}^n)$: We have Parseval's relation:

$$(1.8)\quad \int_{\mathbb{R}^n} đx|u^\wedge(x)|^2 = \int_{\mathbb{R}^n} đx|u(x)|^2 \ , \text{ for all } u \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n) \ .$$

Formula (1.8) is easier to prove as the Fourier inversion formula, asserting $u^{\wedge\vee}=u^{\vee\wedge}=u$ for certain u: We may write

$$(1.9)\quad \int_{Q_N} đx\bar{u}^\wedge(x)v^\wedge(x) = \int đξđη\bar{u}(ξ)v(η)\Pi_{j=1}^n \int_{-N}^N e^{ix_j(ξ_j-η_j)} đx_j \ ,$$

assuming that $u,v\in L^1(\mathbb{R}^n)$, with the 'cube' $Q_N=\{|x_j|\le N, j=1,..,N\}$, some integer N>0. Indeed, the interchange of integrals leading to (1.9) is legal, since the integrand is $L^1(Q_N\times\mathbb{R}^n\times\mathbb{R}^n)$.

Note that $\int_{-N}^N e^{ist}dt = 2\frac{\sin sN}{s}$, $s\neq0$, $= 2N$, $s=0$, allowing evaluation of the inner integrals at right of (1.9). With $\int đξđη = \int đξ\int đη$, and $η = ξ - ζ/N$, $đη = N^{-n}đζ$, (1.9) assumes the form

$$(1.10)\quad \int_{Q_N} đx\bar{u}^\wedge(x)v^\wedge(x) = \int đξ\bar{u}(ξ)\int đζ v(ξ-ζ/N)\Pi_{j=1}^n φ(ζ_j) \ ,$$

where $φ(t) = (2/\pi)^{1/2}\frac{\sin t}{t}$, $t\neq0$, continuously extended into $t=0$.

For $v\in C(\mathbb{R}^n)$, as $N\to\infty$, the function $v(ξ-ζ/N)$ will converge to $v(ξ)$, independent of ζ . Thus one expects the inner integral at right of (1.10) to converge to $v(ξ)\int\Pi_{j=1}^n φ(ζ_j)đζ_j = v(ξ)$, since

$$(1.11)\quad \int_0^\infty \sin t\, dt/t = \pi/2 \ .$$

Legalization of this argument will confirm Parseval's relation, since the right hand converges to the right hand side of (1.8), as $N\to\infty$. With $u\in L^1$ and $v\in C_0^\infty$ (setting $φ_n(ζ)=\Pi φ(ζ_j)$) write

$$(1.12)\quad \int đξ\bar{u}(ξ)\int đζφ_n(ζ)(v(ξ-ζ/N)-v(ξ)) = \int_{Q_N} đx\bar{u}^\wedge v^\wedge - \int_{\mathbb{R}^n} đx\bar{u}v \ .$$

To show that the inner integral at left goes to 0 as $N\to\infty$ it is more skilful to use the integration variable $θ=ζ/N$, $dζ=N^n dθ$. For $n=1$, $\int \sin Nθ\ (v(ξ-θ)-v(ξ))dθ/θ = \int_{|θ|\le δ} + \int_{|θ|\ge δ} = I_0 + I_\infty$.

Here we get (with $w(θ) = (v(ξ-θ)-v(ξ))/θ$)

$$|I_0| \le δ\|v'\|_{L^\infty},\quad I_\infty=\tfrac{1}{N}((w(θ)\cos(Nθ)|_{θ=-δ}^{θ=δ} + \int_{|θ|\ge δ}\cos(Nθ)w_{|θ}(θ)dθ).$$

The latter gives $I_\infty \le \frac{c}{Nδ}(\|v\|_{L^\infty}+\|v'\|_{L^\infty})$, with a constant c, only depending on the volume of supp v, i.e., it is fixed after fixing v . The estimates imply the inner integral to go to 0, uniformly as $x\in\mathbb{R}^n$. For $u\in L^1$ the Lebesgue theorem then implies the left hand side of (1.12) to tend to 0, as $N\to\infty$, for each fixed $v\in C_0^\infty$.

For general n the proof is a bit less transparent, but remains the same: Split the inner integral into a sum of integrals over a small neighbourhood of 0 and its complement. In the first term use differentiability of v; in the second an integration by parts.

We now have a 'polarized' Parseval relation, in the form

$$(1.13)\qquad \int_{\mathbb{R}^n} đx\,\bar{u}^\wedge v^\wedge = \int_{\mathbb{R}^n} đx\,\bar{u}v\ ,\ \text{for } u\in L^1\ ,\ v\in C_0^\infty\ .$$

For $u\in L^1\cap L^2$ pick a sequence $u_j\in C_0^\infty$ with $\|u-u_j\|_{L^1}\to 0$, $\|u-u_j\|_{L^2}\to 0$, as is possible. Then, since $u_j-u_l\in C_0^\infty\subset L^2$, (1.13) with $u=v=u_j-v_j$ implies $\|u_j^\wedge-u_l^\wedge\|_{L^2}=\|u_j-u_l\|_{L^2}\to 0$, $j,l\to\infty$. In other words, u_j and $u_j^\wedge$ both converge in L^2. Clearly, $u_j^\wedge\to u^\wedge$. Indeed, initially we showed uniform convergence over $\mathbb{R}^n$, while the L^2-limit $z=\lim u_j^\wedge$ satisfies $(u^\wedge,\varphi)=\int\bar{z}\varphi dx$ for all $\varphi\in C_0^\infty$. This yields $\int(u^\wedge-z)\varphi dx=0$ for all such φ, hence $u^\wedge=z$ (almost everywhere), since C_0^∞ is dense in L^2. Substituting $u=v=u_j$ in (1.13), letting $j\to\infty$, it follows that (1.8) is valid for all $u\in L^1\cap L^2$, confirming Parseval's relation.

Clearly (1.13) also holds for all $u,v\in L^1\cap L^2$. We use it to prove the Fourier inversion Let n=1. For $v\in L^1\cap L^2$, $u=\chi_{[0,x_0]}$, some $x_0>0$ apply (1.13). Confirm by calculation of the integral that

$$(1.14)\qquad (2\pi)^{1/2}u^\wedge(x) = (e^{-ixx_0}-1)/(-ix) = h_{x_0}(x)\ ,\quad x\neq 0\ ,$$

hence

$$(1.15)\qquad \int_0^{x_0} v(x)dx = \int đx\,v^\wedge(x)\bar{h}_{x_0}(x)dx\ .$$

The Fourier inversion formula is a matter of differentiating (1.15) for x_0 under the integral sign, assuming that this is legal Consider the difference quotient:

$$(1.16)\qquad (2\delta)^{-1}\int_{x_0-\delta}^{x_0+\delta} v(x)dx = \int đx\,v^\wedge(x)e^{ixx_0}\,\frac{\sin\delta x}{\delta x}\ .$$

Assuming only that v, $v^\wedge$ both are in L^1, it follows indeed that

$$(1.17)\ \lim_{\delta\to 0}(2\delta)^{-n}\int_{Q_{x_0,\delta}} v(x)dx = \int đx\,v^\wedge(x)e^{ixx_0} = (v^\wedge)^\vee(x_0),\ x_0\in\mathbb{R}^n.$$

(Actually, our proof works for n=1, $x_0>0$ only, but can easily be extended to all x_0, and general n. One must replace the derivative d/dx_0 by a mixed derivative $\partial^n/(\partial x_{0\,1}\ldots\partial x_{0\,n})$.) Indeed, letting $\delta\to 0$ in (1.17) we obtain (1.15), using that $\sin(\delta x)/(\delta x) \to 1$ uniformly on compact sets, and boundedly on $\mathbb{R}$, as $\delta\to 0$.

If v is continuous at x_0 then clearly the left hand side of (1.17) equals $v(x_0)$, giving the Fourier inversion formula, as it is well known. For $n=1$, if v has a jump at x_0 then the left hand side of (1.17) equals the mean value $(v(x_0+0)+v(x_0-0))/2$.

Again for $n=1$ a limit of (1.16), as $\delta\to 0$ exists, if only

$$(1.18)\qquad \lim_{\alpha\to\infty}\int_{-\alpha}^{+\alpha} v^\wedge(x)\,đx \quad ,$$

the principal value, exists (cf. pbm.6), without requiring $v^\wedge \in L^1$.

We summarize our results thus far:

<u>Proposition 1.1</u>. The Fourier transform $u^\wedge$ of (1.1) and its complex conjugate $u^\vee = (\bar{u}^\wedge)^-$ are defined for all $u \in L^1(\mathbb{R}^n)$, and we have $u^\wedge, u^\vee \in CO(\mathbb{R}^n)$. For $u \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ we have Parseval's relation (1.8). If both $u \in L^1(\mathbb{R}^n)$ and $u^\wedge \in L^1(\mathbb{R}^n)$ hold, then we have $u^{\wedge\vee}(x) = u^{\vee\wedge}(x) = u(x)$ for almost all $x \in \mathbb{R}^n$.

It is known that the Banach space $L^1(\mathbb{R}^n)$ is a commutative Banach algebra under the convolution product

$$(1.19)\qquad u*v = w \;,\; w(x) = \int đy\,u(x-y)w(y) = \int đy\,v(x-y)u(y) \;.$$

Indeed,

$$(1.20)\qquad \|w\|_{L^1} = \int |w(x)|dx \le \kappa_n \int dx \int dy\,|u(x-y)||v(y)| = \kappa_n \|u\|_{L^1}\|v\|_{L^1} \;.$$

Prop.1.2, below, clarifies the role of the Fourier transform F for this Banach-algebra: F provides the Gelfand homomorphism.

<u>Proposition 1.2</u>. For $u,v \in L^1$ let $w = u*v$. Then we have

$$(1.21)\qquad w^\wedge(\xi) = u^\wedge(\xi)v^\wedge(\xi) \;,\; \xi \in \mathbb{R}^n \;.$$

<u>Proof</u>. We have

$$w^\wedge(\xi) = \int đx\,e^{-ix\xi}\int đy\,u(x-y)v(y) = \int đy\,e^{-iy\xi}\int dx\,u(x-y)e^{-i(x-y)\xi} \;.$$

The substitution $x-y=z$, $dy=dz$ thus confirms (1.21), q.e.d.

The importance of the Fourier transform for PDE's hinges on

<u>Proposition 1.3</u>. If $u^{(\beta)} \in L^1$ for all $\beta \le \alpha$ then

$$(1.22)\qquad u^{(\alpha)\wedge}(\xi) = i^{|\alpha|}\xi^\alpha u^\wedge(\xi) \;,\; \xi \in \mathbb{R}^n \;.$$

Proof. Partial integration gives $\int dx e^{-ix\xi} u^{(\alpha)}(\xi) = (-1)^{|\alpha|} \int dx \partial_x^\alpha (e^{ix\xi})$ (with vanishing boundary integrals), implying (1.21), q.e.d.

Given a linear partial differential equation

$$(1.23) \qquad P(D)u = f \quad , \; P(D) = \sum_{|\alpha|\le N} a_\alpha D_x^\alpha ,$$

where $f \in L^1(\mathbb{R}^n)$, $D_{x_j} = -i\partial_{x_j}$, one might attempt to find solutions by taking the Fourier transform. With proper assumptions (1.21) gives

$$(1.24) \qquad P(\xi)u^\wedge(\xi) = f^\wedge(\xi) \; .$$

Assuming that $e = (\frac{1}{P(x)})^\vee$ exists, (1.24) will assume the form

$$(1.25) \qquad u^\wedge(\xi) = e^\wedge(\xi) f^\wedge(\xi) \; ,$$

which by prop.1.2 (and Fourier inversion) is equivalent to

$$(1.26) \qquad u(x) = \int đy\, e(x-y) f(y) \; .$$

Presently, (1.26) can only have a formal meaning, since normally $(1/P) \notin L^1$, or $f \notin L^1$, or $u \notin L^1$, in practical applications.

However, as to be discussed in the sections below, the Fourier transform may be extended to more general classes of functions and to generalized functions. Then (1.26) yields a powerful tool for solving problems in constant coefficient PDE's (cf. sec.4).

Problems. 1) For n=1 obtain the Fourier transforms of the functions a) $(a^2+x^2)^{-1}$, $a>0$; b) $(\sin^2 ax)/x^2$, $a>0$; c) $1/\cosh x$.

2) For general n obtain the Fourier transform of $e^{-\alpha x^2}$, $\alpha>0$.
3) Obtain the Fourier transform of $f(x) = (1+|x|^2)^{-\nu}$, where $\nu>n/2$ (Hint: A knowledge of Bessel functions is required for this problem). 4) Construct a function $f(x) \in L^1(\mathbb{R}^n)$ such that $f^\wedge \notin L^1$.
5) The Riemann-Lebesgue lemma states that $f^\wedge \in CO$ whenever $f \in L^1$. Is it true that even $\hat{f}(x) = O(\langle x\rangle^{-\varepsilon})$ for each $f \in L^1$ with some $\varepsilon>0$? 6) Combining some facts, derived above, show that, for n=1, every piecewise smooth function $f(x) \in L^1(\mathbb{R})$ has a Fourier transform satisfying $f(x) = O(1/x)$, as $|x|$ is large, and satisfying

$$(1.27) \qquad (f(x+0)+f(x-0))/2 = \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} đy\, e^{ixy} f^\wedge(y) \; , \; x \in \mathbb{R} \; .$$

Here 'piecewise smooth' means, that $\mathbb{R}$ may be divided into finitely many closed subintervals in each of which f is C^1 , possibly after changing its value at boundary points.

2. Fourier analysis for temperate distributions on $\mathbb{R}^n$.

We assume that the reader is familiar with the concept of distribution, as a continuous linear functional on the space $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$. A linear functional $f:\mathcal{D} \to \mathbb{C}$ is said to be continuous if $\langle f,\varphi_j\rangle \to 0$ whenever $\varphi_j \to 0$ in $\mathcal{D}$. The latter means that (i) $f_j \in \mathcal{D}$, $j=1,2,\dots$, (ii) $\operatorname{supp} \varphi_j \in K \subset\subset \mathbb{R}^n$, K independent of j, (iii) $\sup\{|\varphi^{(\alpha)}(x)|: x\in \mathbb{R}^n\} \to 0$, as $j\to\infty$, for every α. The space of distributions on $\mathbb{R}^n$ is called $\mathcal{D}'=\mathcal{D}'(\mathbb{R}^n)$. The space $L^1_{loc}(\mathbb{R}^n)$ of locally integrable functions is naturally imbedded in $\mathcal{D}'$ by defining

$$(2.1) \qquad \langle f,\varphi\rangle = \int f(x)\varphi(x)dx \ , \text{ for } f\in L^1_{loc} .$$

The derivatives $f^{(\alpha)}=\partial_x^\alpha f$ of a distribution $f \in \mathcal{D}'$ are defined by

$$(2.2) \qquad \langle f^{(\alpha)},\varphi\rangle = (-1)^{|\alpha|}\langle f,\varphi^{(\alpha)}\rangle \ , \ \varphi \in \mathcal{D} ,$$

the product of a distribution $f \in \mathcal{D}'$ and a $C^\infty(\mathbb{R}^n)$ function g by

$$(2.3) \qquad \langle gf,\varphi\rangle = \langle f,g\varphi\rangle \ , \ \varphi \in \mathcal{D} .$$

Thus Lf is defined for any distribution $f\in \mathcal{D}'(R^n)$ and linear differential operator $L=\sum a_\alpha \partial_x^\alpha$ with coefficients $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$.

While the value f(x) of a distribution at a point x is a meaningless concept, one may talk about the restriction $f|\Omega$ of $f\in \mathcal{D}'(\mathbb{R}^n)$ to an open subset Ω , and its properties: First of all, the space $\mathcal{D}'(\Omega)$ of distributions over Ω consists of the continuous linear functionals on $\mathcal{D}(\Omega)=C_0^\infty(\Omega)$, with continuity defined as for $\mathbb{R}^n$. For $f\in \mathcal{D}'(\mathbb{R}^n)$, the restriction $f|\mathcal{D}(\Omega)$ defines a distribution of $\mathcal{D}'(\Omega)$, denoted by $f|\Omega$. Thus, for example, it is meaningful to say that $f\in \mathcal{D}'(\mathbb{R}^n)$ is a function (a $C^k(\Omega)$-function, etc.) in an open set $\Omega\subset \mathbb{R}^n$ - it means that $f|\Omega$ has this property. For a distribution $f\in \mathcal{D}'(\Omega)$ on an open set the derivatives and product with $g\in C^\infty(\Omega)$ is defined as in (2.2) , (2.3) . The support supp f (singular support sing supp f) of $f\in \mathcal{D}'$ is defined as the smallest closed set E (intersection of all closed sets E) such that f=0 (such that f is C^∞) in the complement of E .

The concept of Fourier transform can be generalized to distributions on $\mathbb{R}^n$, with multiple benefit: Some non-L^1-functions will get distributions as Fourier transforms. Certain distributions will get functions as Fourier transforms. The Fourier inversion formula and many assumptions (limit interchanges) will simplify.

We used the Fourier integral of (1.1) only for $u\in L^1(\mathbb{R}^n)$. It is practical to introduce a growth restriction for $u\in \mathcal{D}'(\mathbb{R}^n)$ if we want $u^\wedge$ to be a distribution again. Later on (sec.3) we also define $u^\wedge$ for general $u\in \mathcal{D}'(\mathbb{R}^n)$, but it no longer will be a distribution in $\mathcal{D}'(\mathbb{R}^n)$. We follow [Schw$_1$] here, but [GS] in sec.3.

The growth restriction is imposed by requesting that $u\in \mathcal{D}'$ allows an extension to a larger space of testing functions called $\mathcal{S}$. Here $\mathcal{S}$ - the space of rapidly decreasing functions- consists of all $\varphi\in C^\infty(\mathbb{R}^n)$ such that for all multi-indices α and $k=1,2,\ldots$,

$$\varphi^{(\alpha)}(x) = O(\langle x\rangle^{-k}) . \tag{2.4}$$

- the derivatives of φ decay faster than every power $\langle x\rangle^{-k}$.

Note that, equivalently, we could have prescribed that for every α one (and the same) of the following conditions be satisfied:

$$\langle x\rangle^k u^{(\alpha)}(x) \text{ (for every } k=1,2,..), \text{ or } x^\beta u^{(\alpha)}(x) \text{ (for every } \beta),$$

$$\text{or } (x^\beta u(x))^{(\alpha)} \text{ (for every } \beta), \text{ is } O(1) \text{ , or is } o(1) \text{ , or is} \tag{2.5}$$

$$CB \text{ , or } CO \text{ , or } L^2 \text{ , or } L^p \text{ (for some } 1\le p\le\infty) .$$

Indeed, for a given α one of these conditions may be weaker or stronger than the other. However for <u>all</u> α simultaneously all conditions are equally strong. One must use Leibniz' formula to handle interchanges of ∂_x^α and multiplications (cf. lemma 2.8).

The above at once gives the following:

<u>Theorem 2.1</u>. We have $\mathcal{S}\subset L^1(\mathbb{R}^n)$, so that $u^\wedge$ of (1.1) (and $u^\vee$) are defined on $\mathcal{S}$. Moreover, for $u\in \mathcal{S}$, we have $u^\wedge, u^\vee\in \mathcal{S}$, and

$$(u^\wedge)^\vee(x) = (u^\vee)^\wedge(x) = u(x) , \ x \in \mathbb{R}^n . \tag{2.6}$$

The Fourier transform and its conjugate therefore define bijective linear maps $\mathcal{S} \leftrightarrow \mathcal{S}$, inverting each other.

<u>Proof</u>. Using repeated partial integration and $x^\alpha e^{-ix\xi}=i^{|\alpha|}\partial_\xi^\alpha e^{-ix\xi}$, $\partial_x^\beta e^{-ix\xi}=i^{-|\beta|}\xi^\beta e^{-ix\xi}$ get $\int dx e^{-ix\xi}x^\alpha u^{(\beta)}(x)=i^{|\alpha|}\partial_\xi^\alpha\int dx e^{-ix\xi}u^{(\beta)}(x)$

$=i^{|\alpha|+|\beta|}\partial_\xi^\alpha\xi^\beta\int dx e^{-ix\xi}u(x)$, hence

$$(x^\beta u^{(\alpha)})^\wedge(\xi) = i^{|\alpha|+|\beta|}(\xi^\alpha u^\wedge(\xi))^{(\beta)} . \tag{2.7}$$

In fact, we get $x^\beta u^{(\alpha)} \in L^1$, for every α,β , by the equivalence