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Australian Mathematical Society Lecture Series. 5

2-Knots and their Groups

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CAMBRIDGE UNIVERSITY PRESS

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New York New Rochelle Melbourne Sydney

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Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521371735

First published 1989
Re-issued in this digitally printed version 2008

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-37173-5 hardback
ISBN 978-0-521-37812-3 paperback

Cambridge University Press
978-0-521-37812-3 - 2-Knots and their Groups
Jonathan Hillman
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Pour les noeuds de S^2 en S^4 on ne sait pas grand chose

Preface

Since Gramain wrote the above words in a Seminaire Bourbaki report on classical knot theory in 1976 there have been major advances in 4-dimensional topology, by Casson, Freedman and Quinn. Although a complete classification of 2-knots is not yet in sight, it now seems plausible to expect a characterization of knots in some significant classes in terms of invariants related to the knot group. Thus the subsidiary problem of characterizing 2-knot groups is an essential part of any attempt to classify 2-knots, and it is the principal topic of this book, which is largely algebraic in tone. However we also draw upon 3-manifold theory (for the construction of many examples) and 4-dimensional surgery (to establish uniqueness of knots with given invariants). It is the interplay between algebra and 3- and 4-dimensional topology that makes the study of 2-knots of particular interest.

Kervaire gave homological conditions which characterize high dimensional knot groups and which 2-knot groups must satisfy, and showed that any high dimensional knot group with a presentation of deficiency 1 is a 2-knot group. Bridging the gap between the homological and combinatorial conditions appears to be a delicate task. For much of this book we shall make a further algebraic assumption, namely that the group have an abelian normal subgroup of rank at least 1. This is satisfied by the groups of many fibred 2-knots, including all spun torus knots and cyclic branched covers of twist spun knots. The evidence suggests that if the abelian subgroup has rank at least 2 then the group is among these, and the problem is then related to that of characterizing 3-manifold groups and their automorphisms. Most known knots with such groups can be characterized algebraically, modulo the s-cobordism theorem. However in the rank 1 case there are examples which are not the groups of fibred knots, and here less is known.

The other class of groups that is of particular interest as it contains the groups of (spun) classical knots consists of those which have cohomological dimension 2 and deficiency 1. (If some standard conjectures hold these conditions are equivalent for knot groups). One striking member of this class is the group Φ with presentation $\langle a,t \mid tat^{-1} = a^2 \rangle$, whose commutator subgroup is a torsion free rank 1 abelian group. All other knot groups with deficiency 1 and nontrivial torsion free abelian normal subgroups are iterated free products of torus knot groups, amalgamated over central copies of Z, and are the groups of fibred 2-knots. We show that any knot with such a group (and more generally, whose group has free commutator subgroup) can be characterized algebraically, modulo the s-cobordism theorem. Together these two classes contain the groups of the most familiar and important examples of 2-knots. However we have by no means completed their classification, and the problem of organizing the groups outside these classes remains quite open. (The formation of sums and satellites should play a part here).

We shall now outline the chapters in somewhat greater detail. In Chapter 1 we give the basic definitions and background results on the geometry of knots and we show how the classification of higher dimensional knots can be reduced (essentially) to the classification of the closed manifolds built from the ambient spheres by surgery on such knots. As far as possible these definitions and results have been formulated so as to apply in all dimensions. We have chosen to work in the TOP category as our chief interest is in the 4-dimensional case, where PL or (equivalently) DIFF techniques are not yet adequate.

In Chapter 2 we give Kervaire's characterization of high dimensional knot groups, and variations on this theme: link groups, commutator subgroups of knot groups, centres of knot groups. We also give his partial results on 2-knot groups. Counter examples to show that not all high dimensional knot groups can be 2-knot groups were found independently by various people; most of their arguments used duality in the infinite cyclic cover of the exterior of the knot. We review some of these arguments, and we show that the exterior of a nontrivial n-knot with $n > 1$ is never aspherical, giving Eckmann's proof via duality in the universal cover.

Chapter 3 contains our key result. We show that in contrast to the theorem of Dyer-Vasquez and Eckmann just quoted the closed 4-manifold obtained by surgery on a 2-knot is often aspherical. If T is the maximal locally-finite normal subgroup of a 2-knot group π and π/T has an abelian normal subgroup of rank 1 such that the quotient has finitely many ends and if a further, technical condition (that may prove to be redundant) holds then either π' is finite or $\pi/T = \Phi$ or π/T is an orientable Poincaré duality group over Q of formal dimension 4. The latter is also true if π/T has an abelian normal subgroup of rank greater than 1.

In the next three chapters we examine these cases separately. In Chapter 4 we determine the 2-knot groups with finite commutator subgroup. All of these can be realized by fibred 2-knots, and many by twist spun classical knots. We show also that if $\pi/T = \Phi$ and T is nontrivial then it must be infinite; in fact we believe that in this case T must be trivial. In Chapters 5 and 6 we consider the Poincaré duality cases. Here there is a further subdivision of cases, according to the rank of the abelian normal subgroup (which must be at most 4). All the known examples with a torsion free abelian normal subgroup of rank 2 derive from twist spun torus knots. The groups of aspherical Seifert fibred 3-manifolds may be characterized as PD_3-groups which have subgroups of finite index with nontrivial centre and infinite abelianization. Using this, we give an algebraic characterization of the groups of 2-knots which are cyclic branched covers of twist spins of torus knots.

In Chapter 6 we determine the 2-knot groups with abelian normal subgroups of rank greater than 2, and the results of these three chapters are combined to show that if π has an ascending series whose factors are locally-finite or locally-nilpotent, then it is in fact locally-finite by solvable. If moreover π has an abelian normal subgroup of positive rank then it is finite by solvable, and we describe all such groups. (We doubt that there are any other 2-knot groups with such ascending series).

In the last two chapters we attempt to recover 2-knots from group theoretic invariants. As we observe in Chapter 1 a knot K is determined up to changes of orientation and "Gluck reconstruction" by a certain closed 4-manifold $M(K)$ together with a conjugacy class in the knot group $\pi_1(M)$. We first try to determine the homotopy type of M in terms of algebraic invariants. The problem of the homeomorphism type may then be reduced to standard questions of surgery. For knots whose group is torsion free and polycyclic we are completely successful, for the surgery techniques are then available to solve the problem. We show also that if the commutator subgroup is an infinite, nonabelian nilpotent group then, excepting for two such groups, the knot is determined up to inversion by its group alone.

Freedman has shown that surgery techniques apply whenever the group is as in Chapter 6, but in general it is difficult to compute the obstructions. On the other hand, for many fibred 2-knots we can determine the (simple) homotopy type and show that the surgery obstructions are 0, but it is not yet known whether 5-dimensional s-cobordisms with such groups are always products.

After Chapter 8 there are two appendices. The first considers the 4-dimensional geometries that can be supported by some $M(K)$. (Among these are some complex surfaces). In the second it is shown that certain Cappell-Shaneson 2-knots are reflexive if and only if every totally positive unit in the cubic number field generated by a root of the Alexander polynomial is a square in that field. After there is a list of open questions on 2-knots and related topics. Some of these are well known and very difficult; others are more technical and algebraic, but have also resisted solution so far.

As the algebra used in this book may perhaps be unfamiliar for many topologists we would like to stress here that our principal references have been the Queen Mary College lecture notes of Bieri for homological group theory, and the text of Robinson for other aspects of group theory.

I would like to thank William Dunbar, Ross Geoghegan, John Groves, Laci Kovács, Peter Kropholler, Darryl McCullough, Mike Mihalik, Peter M. Neumann, Steve Plotnick, Peter Scott and Shmuel Weinberger for their correspondance and advice on various aspects of this work. I would also like to acknowledge the support of the U.K. Science and Engineering Research Council (as a Visiting Fellow at the University of Durham), which enabled me to meet Peter Kropholler and Peter Scott, and of the Australian Research Grants Scheme, for a grant which brought Steve Plotnick to Macquarie University in mid-1987.

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