CAMBRIDGE

Cambridge University Press
978-0-521-37812-3 - 2-Knots and their Groups
Jonathan Hillman
Excerpt
More information

Chapter 1 KNOTS AND RELATED MANIFOLDS

In this chapter are the basic definitions and constructions of the objects that we shall study. In particular we show how the classification of higher dimensional knots can be reduced (essentially) to the classification of certain closed manifolds. We also give a number of results on the geometry of these objects, for the most part without proof, as we shall not use them in a crucial way in our arguments later (which are primarily algebraic). We shall first state some of our conventions on notation and terminology.

Let $D^n = \{ \langle x_1,\dots x_n\rangle \text{ in } R^n \mid \Sigma x_i^2 \leqslant 1 \}$ be the n-disc and let $S^n = \partial D^{n+1}$ be the n-sphere. The standard orientation of R^n induces an orientation of D^n, and hence one of S^{n-1} by the convention that the boundary of an oriented manifold be oriented compatibly with taking the inward normal last (cf. [RS: pages 44–45]). We shall always assume that these standard discs and spheres have been given such standard orientations.

An *embedding* is a 1–1 map which is a homeomorphism onto its image. All isotopies of locally flat embeddings of manifolds in manifolds shall be *ambient* isotopies. The inclusion of R^{n+1} into R^{n+2} as the hyperplane defined by the equation $x_{n+2} = 0$ induces the *equatorial* embedding $e_n: S^n \to S^{n+1}$. The interior of a subset N of a topological space shall be denoted by *int N*.

The expression $A \cong B$ means that the objects A and B are isomorphic in some category appropriate to the context. When there is a canonical isomorphism, or after a particular isomorphism has been chosen, we shall write $A = B$. (For instance the fundamental group of a circle is infinite cyclic, and choosing an isomorphism with Z corresponds to choosing an orientation for the circle). Qualifications shall usually be omitted when there is no risk of ambiguity. In particular, we may abbreviate $X(K)$, $M(K)$ and πK as X, M and π respectively.

Knots

An n-*knot* is a locally flat embedding $K: S^n \to S^{n+2}$. (We shall also use the terms "classical knot" when $n = 1$, "higher dimensional knot" when $n \geqslant 2$ and "high dimensional knot" when $n \geqslant 3$). Such a knot is determined up to isotopy by its image $K(S^n)$, considered as an oriented codimension 2 submanifold of S^{n+2}, and so we may let K also denote this submanifold. Two n-knots K_0 and K_1 are isotopic if (and only if) there is an orientation preserving self homeomorphism h of S^{n+2} such that $hK_0 = K_1$, for such a map h is isotopic to the identity [KS: page 292]. Thus for instance if r_n is an orientation reversing self homeomorphism of S^n the knots K, $rK = r_{n+2}K$, $K\rho = Kr_n$ and $-K = rK\rho$ may all be distinct. (Note that the images of K and $K\rho$ are the same, considered as unoriented submanifolds). The knot K is *invertible*, *+amphicheiral* or *−amphicheiral* if it is isotopic to $K\rho$, rK or $-K$ respectively. An n-knot is *trivial* if it is isotopic to the unknot $e_{n+1}e_n$. By the uniqueness of discs an n-knot is trivial if and only if it bounds a locally flat $(n+1)$-disc in S^{n+2}.

If $n \geqslant 4$ then each n-knot is isotopic to a PL n-knot which is unique up to PL (ambient) isotopy, for then $H^i(S^{n+2},K;Z/2Z) = 0$ for $i = 3$ and 4, so the Kirby-Siebenmann obstructions to existence and uniqueness of PL triangulations vanish [KS: Essay IV.10]. All the examples of 2-knots that we shall construct below shall be PL with respect to some triangulation of S^4. However as it is not yet known whether all triangulations of S^4 are PL equivalent, and as the 4-dimensional surgery techniques that we wish to use to characterize certain 2-knots have only been established in the TOP category (and then only for a restricted class of fundamental groups), the above definition of knot seems most suitable.

The exterior

Every (locally flat) n-knot with $n = 2$ is flat [KS 1975] and so there is an embedding $j:S^n\times D^2 \to S^{n+2}$ onto a closed neighbourhood N of K, such that $j|S^n\times\{0\} = K$ and $\partial N = j(S^n\times S^1)$ is bicollared in S^{n+2}. We may assume that j is orientation preserving, and then it is unique up to isotopy *rel* $S^n\times\{0\}$. These results (and their proofs) remain valid when $n = 2$, by Quinn's solution of the 4-dimensional annulus conjecture, together with surgery over Z (cf. [Qu 1982] and [FQ]).

The *exterior* of K is the compact $(n+2)$-manifold $X(K) = S^{n+2}-int\,N$, which is well defined up to homeomorphism as N is unique up to isotopy *rel* K. It inherits an orientation from S^{n+2}, and has boundary homeomorphic to $S^n \times S^1$. The interior of X is homeomorphic to the knot complement $S^{n+2}-K$. The *knot group* is $\pi K = \pi_1(X(K))$. By general position, every element of πK can be represented by an oriented simple closed curve in X, and if $n \geqslant 2$ each conjugacy class in πK (i.e. free homotopy class of maps from S^1 to X) corresponds to an unique isotopy class of such curves. In particular, any oriented simple closed curve isotopic to the oriented boundary of a transverse disc (i.e. to $\{j(s)\} \times S^1$) is called a *meridian* of K, and we shall also use this term to denote the corresponding elements of πK.

By Alexander duality $H_i(X;Z) \cong Z$ if $i = 0$ or 1 and is 0 otherwise. The meridians are all homologous and generate $H_1(X;Z)$ (by the Mayer-Vietoris sequence for $S^{n+2} = X \cup S^n \times D^2$) and so determine a canonical isomorphism $H_1(X;Z) = Z$.

The exterior of a trivial n-knot is homeomorphic to $D^{n+1} \times S^1$. Conversely if $X(K)$ is homotopy equivalent to S^1 then K is trivial [Pa 1957, St 1963, Le 1965, Sh 1968, Fr 1983]. For if γ is a meridian in the interior of X, it has a product neighbourhood U (as X is orientable) and if $X \sim S^1$ then $X - int\, U$ is an s-cobordism and so a product, provided $n \geqslant 3$. It is then easy to see that K bounds a disc in S^{n+2}. (This argument works also in the PL and DIFF categories). This criterion for triviality is also correct when $n = 1$ [Pa 1957] and $n = 2$ [Fr 1983], although the proofs are different. (Note that there may be PL 2-knots with group Z which are *not* PL isotopic to the unknot). The assumption on X can be weakened considerably (see below); in particular if $n = 1$ or 2 an n-knot is trivial if and only if its group is infinite cyclic.

Gluck showed that when $n = 2$ the group of pseudoisotopy classes of self homeomorphisms of $S^n \times S^1$ is $(Z/2Z)^3$, generated by reflections in either factor and by the map τ given by $\tau(x,y) = (\theta(y)(x),y)$ for all x in S^n and y in S^1, where $\theta: S^1 \to SO(n+1)$ is an essential map [Gl 1962]. Browder and Kato extended his result to all higher dimensions [Br 1967, Ka 1969]. As any self homeomorphism of $S^n \times S^1$ extends across $D^{n+1} \times S^1$ (provided $n \geqslant 2$), the closed $(n+2)$-manifold $M(K) = X(K) \cup D^{n+1} \times S^1$ obtained from S^{n+2} by surgery on K is also well defined, and it inherits an orientation from S^{n+2} via X. Since up to homotopy X is the complement of $\{0\}\times S^1$ in M, the inclusion of X into M induces an isomorphism $\pi K = \pi_1(M)$, by general position (for $n \geqslant 2$). The Euler characteristic of M is $\chi(M) = \chi(S^{n+2}) - \chi(S^n\times D^2) + \chi(D^{n+1}\times S^1) = 0$. In fact $H_i(M;Z)$ is Z for $i = 0, 1, n+1$ or $n+2$ and is 0 otherwise, as follows easily from the Mayer–Vietoris sequence for $M = X\cup D^{n+1}\times S^1$. (The orientations of S^{n+2} and K determine canonical isomorphisms). We shall in fact study 2-knots K through the corresponding closed oriented 4-manifolds $M(K)$.

There is however an ambiguity when we attempt to recover K from $M(K)$. The cocore $\gamma = \{0\}\times S^1 \subset D^{n+1}\times S^1 \subset M$ of the original surgery is well defined up to isotopy by the conjugacy class of a meridian in $\pi K = \pi_1(M)$. (In fact the orientation of γ is irrelevant for what follows). Its normal bundle is trivial, so γ has a product neighbourhood, P say, and we may assume that $M - int\, P = X$. But there are two essentially distinct ways of identifying ∂X with $S^n\times S^1 = \partial(S^n\times D^2)$, modulo self homeomorphisms of $S^n\times S^1$ that extend across $S^n\times D^2$, provided $n \geqslant 2$. If we reversed the original construction of M we recover $(S^{n+2}, K) = (X\cup_{j\partial} S^n\times D^2, S^n\times\{0\})$. If however we identify $S^n\times S^1$ with X by means of $j\tau$ we obtain a new pair $(\Sigma, K^*) = (X\cup_{j\tau} S^n\times D^2, S^n\times\{0\})$. By van Kampen's theorem Σ is simply connected, and it is easily seen to have the homology of S^{n+2}. Therefore Σ is homeomorphic to S^{n+2}. We may assume that the homeomorphism is orientation preserving. Thus we obtain a new n-knot K^*, which we shall call the *Gluck reconstruction* of K. The knot K is said to be *reflexive* if it is determined as an unoriented submanifold by its exterior, i.e. if K^* is isotopic to K, rK, $K\rho$ or $-K$.

By the work of Browder, Gluck and Kato, if $n \geqslant 2$ there are at most two n-knots (up to change of orientations) with a given exterior, i.e. if there is an orientation preserving homeomorphism from $X(K_1)$ to $X(K)$ then K_1 is isotopic to K, K^*, $K\rho$ or $K^*\rho$. If the homeomorphism also preserves the homology class of the meridians then K_1 is isotopic to K or K^*. (The long-standing conjecture that each classical knot is determined up to orientation by its exterior has finally been confirmed [GL 1988]. The argument involves some of the deepest results of 3-manifold topology). Thus a knot K is determined up to an ambiguity of order at most 2 by $X(K)$, or (if $n \geqslant 2$) equivalently by $M(K)$ together with the conjugacy class of a meridian in πK. Cappell and Shaneson gave the first examples of knots which are not reflexive [CS 1976]. Their method works for each $n \geqslant 2$ provided that certain $(n+1)\times(n+1)$ integral matrices exist; at present such matrices have been found only for $n = 2$, 3, 4 and 5. Gordon gave a different family of examples when $n = 2$, all of which are PL knots with respect to the standard triangulation of S^4 [Go 1976].

Covering spaces and equivariant homology

We shall let $\widetilde{X}(K)$ and $X'(K)$ denote the *universal* and *maximal abelian* covering spaces (respectively) of $X(K)$. Similarly $\widetilde{M}(K)$ and $M'(K)$ are the corresponding covering spaces of $M(K)$. The fundamental group of X' (and of M', provided that $n \geqslant 2$) is the commutator subgroup π' of the knot group, and by the Hurewicz theorem $\pi/\pi' = H_1(X;Z) = Z$. Thus the cover X'/X is also known as the *infinite cyclic* cover of the knot exterior.

The homology and cohomology of such covering spaces are modules over the group ring of the covering group, and satisfy a form of equivariant Poincaré duality. We shall describe this in somewhat greater generality. Let P be a closed orientable m-manifold with fundamental group G. Up to homotopy we may approximate P by a finite cell complex [KS: Essay III.4]. Let H be a normal subgroup of G and let P_H be the corresponding covering space. We may then lift the cellular decomposition of P to an equivariant cellular decomposition of P_H. The cellular chain complex C_* of P_H with coefficients in a commutative ring R is then a complex of left $R[G/H]$-modules with respect to the action of the covering group G/H. Moreover C_* is a complex of free modules, with a finite basis obtained by choosing one lift of each cell of P. The i^{th} *equivariant homology* module of P with coefficients $R[G/H]$ is the left module $H_i(P;R[G/H]) = H_i(C_*)$, which is clearly isomorphic to $H_i(P_H;R)$ as an R-module, with the action of the covering group determining the $R[G/H]$-structure. The i^{th} *equivariant cohomology* module of P with coefficients $R[G/H]$ is the right module $H^i(P;R[G/H]) = H^i(Hom_{R[G/H]}(C_*,R[G/H]))$, which may be interpreted as cohomology of

P_H with compact supports.

If N is a right $R[G]$-module we shall let $\overline{N}$ denote the left module with the same underlying R-module and the conjugate G-action, determined by $g.n = ng^{-1}$ for g in G and n in N.

The equivariant homology and cohomology are related by Poincaré duality isomorphisms $H_i(P;R[G/H]) = \overline{H^{m-i}(P;R[G/H])}$ and by a Universal Coefficient spectral sequence with E_2 term

$$E_2^{pq} = Ext^q_{R[G/H]}(H_p(P;R[G/H]),R[G/H]) \Rightarrow H^{p+q}(P;R[G/H]),$$

in which the differential d_r has bidegree $(1-r,r)$. If J is a normal subgroup of G which contains H there is also a Cartan-Leray spectral sequence relating the homology of P_H to that of P_J, with E^2 term

$$E^2_{pq} = Tor_p^{R[G/H]}(H_q(P;R[G/H]),R[G/J]) \Rightarrow H_{p+q}(P;R[G/J])$$

in which the differential d^r has bidegree $(-r,r-1)$. There are similar definitions and results for manifold pairs $(P,\partial P)$ and for (co)homology with more general coefficients [W]. For more information on these spectral sequences see [McC].

When K is an n-knot, $P = X(K)$ or $M(K)$, $G = \pi K$ and $H = \pi'$, the group ring $Z[\pi/\pi']$ is the ring of integral Laurent polynomials $\Lambda = Z[Z] = Z[t,t^{-1}]$. Since Λ is noetherian the homology and cohomology of a finitely generated free Λ-chain complex are also finitely generated. The augmentation module Z has projective dimension 1 as it has a short free resolution $0 \to \Lambda \to \Lambda \to Z \to 0$. Therefore the Cartan-Leray spectral sequence for the projection of X' onto X (or of M' onto M) reduces to a long exact sequence (the Wang sequence of the map $X' \to X$):

$$\cdots \to H_i(X;\Lambda) \to H_i(X;\Lambda) \to H_i(X;Z) \to H_{i-1}(X;\Lambda) \to \cdots .$$

Since X has the homology of a circle, it follows that all the maps $t-1:H_i(X;\Lambda) \to H_i(X;\Lambda)$ are surjective for $i > 0$. Therefore they are bijective (since the modules are noetherian) and so the homology modules are all torsion Λ-modules. In particular $Hom_\Lambda(H_p(X;\Lambda),\Lambda) = 0$ for all p so the Universal Coefficient spectral sequence collapses to a collection of short exact sequences

$$0 \to Ext^2_\Lambda(H_{p-2}(X;\Lambda),\Lambda) \to H^p(X;\Lambda) \to Ext^1_\Lambda(H_{p-1}(X;\Lambda),\Lambda) \to 0.$$

(There are very similar results for $H_*(M;\Lambda)$).

The infinite cyclic covering spaces X' and M' behave homologically much like $(n+1)$-manifolds with boundary S^n and empty respectively, at least if we use field coefficients [Mi 1968, Ba 1980]. If $H_i(X;\Lambda) = 0$ for $1 \leqslant i \leqslant (n+1)/2$ then X' is acyclic; thus if also $\pi = Z$ then X is homotopy equivalent to S^1 and so K is trivial. All the classifications of higher dimensional knots to date assume that the knot group is Z and that the infinite cyclic cover of the exterior is highly connected. An n-knot K is called *r-simple* if $X'(K)$ is r-connected; if $n \geqslant 3$ and the knot is $[(n-1)/2]$-simple it is *simple*. (The word is used in a different sense in connection with classical knots). Levine classified simple $(2q+1)$-knots with $q \geqslant 2$ by means of Seifert matrices [Le 1970]; this was reformulated (and reproven) in terms of the duality pairing on the middle dimensional homology by Kearton [Ke 1975]. After Kearton and Kojima had each classified some significant subclasses of simple $2q$-knots with $q \geqslant 4$ by means of analogous (but more complicated) pairings, Farber finished this task as an application of his classification of *stable* n-knots (i.e. those which are r-simple for some $r \geqslant (n+1)/3$) in terms of stable homotopy pairings [Fa 1983]. By analogy with results on the rational homotopy type of highly connected manifolds, one might hope that an r-simple n-knot K in the "formal range" with $r \geqslant (n-1)/4$ would be determined up to finite ambiguity by the cohomology algebra $H^*(M(K);\Lambda)$. (Ideally one might hope to classify all 1-simple knots up to finite ambiguity by algebraic invariants, but it is not yet clear what these should be).

When $n = 1$ or 2 it is more profitable to work with the universal cover $\tilde{X}$ (or $\tilde{M}$). When $n = 1$ the universal cover of X is contractible; this is a remarkable result of Papakyriakopoulos [Pa 1957]. In higher dimensions X is aspherical only when the knot is trivial, as we shall see in Chapter 2. Nevertheless under rather mild assumptions on the group of a 2-knot K the closed 4-manifold $M(K)$ *is* aspherical. (See Chapter 3). This is the main reason that we choose to work with $M(K)$ rather than $X(K)$.

Knot sums, factorization and satellites

The *sum* of two n-knots K_1 and K_2 may be defined (up to isotopy) as the n-knot $K_1\#K_2$ obtained as follows. Let $D^n(\pm)$ denote the upper and lower hemispheres of S^n. We may isotope K_1 and K_2 so that each $K_i(D^n(\pm))$ is contained in $D^{n+2}(\pm)$, $K_1(D^n(+))$ is a trivial n-disc in $D^{n+2}(+)$, $K_2(D^n(-))$ is a trivial n-disc in $D^{n+2}(-)$ and $K_1|S^{n-1} = K_2|S^{n-1}$ (as the oriented boundaries of the images of $D^n(-)$). Then we let $K_1\#K_2 = K_1|D^n(-)\cup K_2|D^n(+)$. By van Kampen's theorem (used several times) $\pi(K_1\#K_2) = \pi K_1 *_Z \pi K_2$ where the amalgamating subgroup is generated by a meridian in each knot group. It is not hard to see that $X'(K_1\#K_2)$ is homotopy equivalent to $X'(K_1)\vee X'(K_2)$ and so in particular $\pi'(K_1\#K_2) = \pi'(K_1)*\pi'(K_2)$.

When $n = 1$ this construction corresponds to tying two knots consecutively in the same loop. We say that a knot is *irreducible* if it is not the sum of two nontrivial knots. Schubert showed that each 1-knot has an essentially unique factorization as a sum of irreducible knots (and so irreducible 1-knots are usually called *prime* knots) [Sch 1949]. In higher dimensions this is false in general. However Dunwoody and Fenn have shown that every n-knot with $n \geqslant 3$ admits some finite factorization into irreducible knots, and moreover for each knot there is a bound on the length of such factorizations [DF 1987]. As the only geometric result needed for their argument is a criterion for recognizing the trivial knot (i.e. K is trivial if $X(K) \sim S^1$), it applies also when $n = 2$, by Freedman's Unknotting Theorem. (It is easy to see that if K is a 2-knot with finitely generated commutator subgroup π' then K has a finite factorization into irreducibles, for the Grushko-Neumann theorem places an upper bound on the number of nontrivial free factors of π'. If $K = \#K_i$ is a factorization with more terms than this bound then at least one of the summands K_i must have group Z and so be trivial).

For each $n \geqslant 3$ there are n-knots which have several distinct factorizations into irreducibles [BHK 1981]. Essentially nothing is known about uniqueness (or otherwise) of factorization when $n = 2$.

A more general method of combining two knots is the process of forming satellites. Although this construction arose in the classical case [Sch 1953], where it is intimately connected with the notion of torus decomposition, we shall describe only the higher dimensional version of [Ka 1983]. Let K_1 and K_2 be n-knots (with $n > 1$) and let γ be a simple closed curve in $X(K_1)$, with a product neighbourhood U. Then $S^{n+2}-int\,U$ is homeomorphic to $S^n \times D^2$ and so we may find a homeomorphism h which carries $S^{n+2}-int\,U$ onto a product neighbourhood of K_2. The knot $\Sigma(K_2;K_1,\gamma) = hK_1$ is called the *satellite* of K_1 about K_2 relative to γ. We also call K_2 a *companion* of K_1. If either $\gamma = 1$ or K_2 is trivial then $\Sigma(K_2;K_1,\gamma) = K_1$. The group of a satellite knot may be computed by means of van Kampen's Theorem (cf. Chapter 3).

Fibred knots

A *Seifert hypersurface* for K is a locally flat, oriented codimension 1 submanifold of S^{n+2} with (oriented) boundary K. By a standard argument these always exist. As we shall make little use of Seifert hypersurfaces in this book, we shall only outline the argument briefly. (We shall however use the phrase "Seifert *manifold*" below in the sense of closed 3-manifold foliated by circles). Using obstruction theory, it may be shown that the projection $pr_2 j^{-1}:\partial X(K) \to S^n \times S^1 \to S^1$ extends to a map $p:X \to S^1$ [Ke 1965]. By topological transversality, we may assume that the inverse image $p^{-1}(1)$ is a bicollared, proper codimension 1 submanifold of X [Qu 1982]. The union $p^{-1}(1) \cup j(S^n \times [0,1])$ is then a Seifert hypersurface for K. If a 2-knot has a Seifert surface which is a once-punctured connected sum of lens spaces and copies of $S^1 \times S^2$ then it is reflexive [Gl 1962].

In general there is no canonical choice of a Seifert hypersurface. However there is one important special case. An n-knot K is *fibred* if we may find such a map p which is the projection of a fibre bundle (i.e. if every point of S^1 is a regular value of p). The sphere S^{n+2} is then swept out by copies of the Seifert hypersurface obtained from $p^{-1}(1)$, which are disjoint except at their common boundary K. The bundle is determined by the isotopy class of the characteristic map, which is a self homeomorphism θ of the fibre F of p. Indeed $X(K)$ is the mapping torus $F\times_\theta S^1 = F\times[0,1]/\sim$, where $(f,0) \sim (\theta(f),1)$ for all f in F. The isotopy class of θ is called the (geometric) monodromy of the bundle. It is easy to see that such a map p extends to a fibre bundle projection $q{:}M(K) \to S^1$. The fibre $\hat{F} = F\cup D^{n+1}$ of q is called the *closed fibre* of K, and $M(K)$ is the mapping torus of the closed monodromy. Conversely, if $n \geqslant 2$ and $M(K)$ fibres over S^1 then we may assume that the characteristic map fixes an $(n+2)$-disc pointwise, and we see that K is fibred. (An analogous result is true when $n = 1$ [Ga 1987]). Many of our examples below shall arise as a result of surgery on a simple closed curve in such a mapping torus. (For instance Cappell and Shaneson construct their pairs of distinct n-knots with homeomorphic exteriors by starting with the mapping torus of a self homeomorphism of $(S^1)^{n+1}$).

A 1-knot K is fibred if and only if π' is free [St 1962]. In high dimensions we may apply Farrell's fibration theorem to obtain a criterion for an n-knot with $n \geqslant 4$ to be fibred [Fa 1970]. This applies also when $n = 3$, provided the knot group is in the class of groups for which 4-dimensional surgery and s-cobordism theorems are known (cf. [Fr 1983]). Little is known when $n = 2$. It is conceivable that every 2-knot whose commutator subgroup is finitely generated and torsion free may be fibred. However we shall show in Chapter 8 that if the 3-dimensional Poincaré conjecture is true then there are 2-knots whose commutator subgroup is $Z/3Z$ which are not fibred.

If K_1 and K_2 are fibred then so is their sum, and the closed fibre of $K_1\#K_2$ is the connected sum of the closed fibres of K_1 and K_2. However in the absence of an adequate criterion for a 2-knot to fibre, we do not know whether every summand of a fibred knot is fibred. In view of the unique factorization theorem for oriented 3-manifolds one might hope that there would be a similar theorem for fibred 2-knots. However the fibre of an irreducible 2-knot need not be an irreducible 3-manifold. (For instance a spun trefoil knot is an irreducible fibred 2-knot, but its closed fibre is $S^2\times S^1\# S^2\times S^1$).

No nontrivial 2-knot which is fibred with monodromy of odd order is reflexive [Pl 1986]. (See also [HP 1988]).