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London Mathematical Society Lecture Note Series. 143

# The Ergodic Theory of Discrete Groups

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CAMBRIDGE UNIVERSITY PRESS  
Cambridge  
New York Port Chester Melbourne Sydney

Cambridge University Press  
978-0-521-37674-7 - The Ergodic Theory of Discrete Groups  
Peter J. Nicholls  
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CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9780521376747](http://www.cambridge.org/9780521376747)

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First published 1989  
Re-issued in this digitally printed version 2008

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-37674-7 paperback

Cambridge University Press  
978-0-521-37674-7 - The Ergodic Theory of Discrete Groups  
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*To Trudy*

## PREFACE

The interaction between ergodic theory and discrete groups has a long history and much work was done in this area by Hedlund, Hopf, Myrberg, and others over fifty years ago. During the last ten years there has been a great resurgence of interest in the area and the field is currently very active. Great advances have been made, and the theory now stands as a well developed branch of mathematical research.

The purpose of this book is two-fold. Firstly, we aim to present a connected account from first principles of the classical work in this area. Much of this material dates to the thirties, and some is difficult to locate. We will gather this material in one place with suitable explanations, simplifications, and connections drawn with the more recent body of literature. Our second aim is to present an introduction to the deep and powerful theory of measures on the limit set of a discrete group which has recently been developed by Patterson, Sullivan, and others. This circle of ideas has applications in a wide variety of problems involving discrete groups, and the notion of a measure on the limit set has emerged as one of the most powerful tools in the theory. We start from first principles and give a detailed account of the construction of the measure classes and the related conformal densities. We then consider ergodic results relative to these new measures and include a discussion of applications of these results to Hausdorff dimension of the limit set and estimates on the orbital counting function.

The book assumes a working knowledge of graduate level analysis and topology. Aside from this, every attempt has been made to keep the presentation self contained. Many of the results we give are to be found in the literature and we have attempted to provide the correct attribution. The proofs that are given here have in some cases been taken directly from the source but for the most part have been constructed by combining the ideas of more than one author.

This book grew out of a fascination for the marvelous work of Dennis Sullivan and S.J. Patterson. Their contributions in recent years to the theory of discrete groups have been astounding, and the theory covered in this book represents just one area of their influence. A major proportion of the results we present are due to Sullivan, although this has not always been made explicit.

With the major purpose of providing the introduction and background necessary for an understanding of, and an appreciation for, measures on the limit set of a discrete group, there are many parts of the subject which are not mentioned — indeed to cover them all would occupy several volumes such as this. In particular we have not touched on symbolic dynamics, nor on the connection between our conformal measures and Gibbs measures recently exploited in the beautiful work of Rees. The survey article of Patterson [Patterson, 1987] should be consulted for further information on these topics and for a good bibliography.

There are many individuals who have helped in the preparation of this book. My friend and teacher Alan Beardon has been a constant source of encouragement and support, and has provided crucial assistance at several stages of the project. My colleague Peter Waterman has read the entire manuscript with great care and provided many valuable insights. Lars Ahlfors has graciously permitted me to quote extensively from his beautifully written Minnesota lecture notes. James Norris has been a very supportive Dean, providing me both with facilities and with time for the completion of the project. Thanks are also due to Sara Clayton for help with the preparation of the manuscript. I would like also to express gratitude to the National Science Foundation for their support during the early stages of the project. Of course, I owe a major debt of gratitude to my family who have put up with my constant preoccupation with "the book" over a long period of time.

Peter Nicholls  
DeKalb Illinois  
March 1989

## CONTENTS

### CHAPTER 1

#### Preliminaries

|                                   |    |
|-----------------------------------|----|
| 1.1 Area                          | 1  |
| 1.2 The Hyperbolic Space          | 3  |
| 1.3 Moebius Transforms            | 7  |
| 1.4 Discrete Groups               | 15 |
| 1.5 The Orbital Counting Function | 18 |
| 1.6 Convergence Questions         | 20 |

### CHAPTER 2

#### The Limit Set

|                                 |    |
|---------------------------------|----|
| 2.1 Introduction                | 23 |
| 2.2 The Line Transitive Set     | 25 |
| 2.3 The Point Transitive Set    | 27 |
| 2.4 The Conical Limit Set       | 28 |
| 2.5 The Horospherical Limit Set | 37 |
| 2.6 The Dirichlet Set           | 39 |
| 2.7 Parabolic Fixed Points      | 42 |

### CHAPTER 3

#### A Measure on the Limit Set

|   |    |
|---|----|
| 3.1 Construction of an Orbital Measure                | 45 |
| 3.2 Change in Base Point                              | 49 |
| 3.3 Change of Exponent                                | 51 |
| 3.4 Variation of Base Point and Invariance Properties | 53 |
| 3.5 The Atomic Part of the Measure                    | 57 |

### CHAPTER 4

#### Conformal Densities

|                      |    |
|----------------------|----|
| 4.1 Introduction     | 68 |
| 4.2 Uniqueness       | 70 |
| 4.3 Local Properties | 71 |



|                                   |                                      |     |
|-----------------------------------|--------------------------------------|-----|
| 4.4                               | The Conical Limit Set                | 75  |
| 4.5                               | The Orbital Counting Function        | 79  |
| 4.6                               | Convex Co-Compact Groups             | 81  |
| 4.7                               | Summary                              | 87  |
| <b>CHAPTER 5</b>                  |                                      |     |
| Hyperbolically Harmonic Functions |                                      |     |
| 5.1                               | Introduction                         | 89  |
| 5.2                               | Harmonic Measure                     | 93  |
| 5.3                               | Eigenfunctions                       | 96  |
| <b>CHAPTER 6</b>                  |                                      |     |
| The Sphere at Infinity            |                                      |     |
| 6.1                               | Introduction                         | 100 |
| 6.2                               | Action on $S$                        | 102 |
| 6.3                               | Action on $S \times S$               | 105 |
| 6.4                               | Action on Other Products             | 109 |
| <b>CHAPTER 7</b>                  |                                      |     |
| Elementary Ergodic Theory         |                                      |     |
| 7.1                               | Introduction                         | 113 |
| 7.2                               | The Continuous Case                  | 116 |
| 7.3                               | Invariant Measures                   | 126 |
| <b>CHAPTER 8</b>                  |                                      |     |
| The Geodesic Flow                 |                                      |     |
| 8.1                               | Definition                           | 128 |
| 8.2                               | Basic Transitivity Properties        | 135 |
| 8.3                               | Ergodicity                           | 140 |
| <b>CHAPTER 9</b>                  |                                      |     |
| Geometrically Finite Groups       |                                      |     |
| 9.1                               | Introduction                         | 147 |
| 9.2                               | Volume of the Line Element Space     | 152 |
| 9.3                               | Hausdorff Dimension of the Limit Set | 154 |
| <b>CHAPTER 10</b>                 |                                      |     |
| Fuchsian Groups                   |                                      |     |
| 10.1                              | Introduction                         | 161 |
| 10.2                              | The Upper Half-Plane                 | 162 |
| 10.3                              | Geodesic and Horocyclic Flows        | 170 |
| 10.4                              | The Unit Disc                        | 174 |

Cambridge University Press  
978-0-521-37674-7 - The Ergodic Theory of Discrete Groups  
Peter J. Nicholls  
Frontmatter  
[More information](#)

---

|                              |            |
|------------------------------|------------|
| <b>Contents</b>              | <b>xi</b>  |
| 10.5 Ergodicity and Mixing   | 182        |
| 10.6 Unique Ergodicity       | 189        |
| 10.7 A Lattice Point Problem | 199        |
| <b>REFERENCES</b>            | <b>209</b> |
| <b>INDEX OF SYMBOLS</b>      | <b>215</b> |
| <b>INDEX</b>                 | <b>216</b> |