CHAPTER 1

Preliminaries

1.1 Area

One of our main concerns in this book is with the measure of certain subsets of the unit sphere. Accordingly, in this section, we introduce Lebesgue surface area on the unit sphere in R^n.

Throughout we denote by B the unit ball in R^n and by S the unit sphere. Thus, denoting points of R^n by $x = (x_1, x_2, \ldots, x_n)$ and writing $|x| = \left(\sum x_i^2\right)^{1/2}$ we have

$$B = \{x : |x| < 1\} \quad \text{and} \quad S = \{x : |x| = 1\}.$$

Note that we will throughout write vectors as rows, but in operations involving matrices, they will be interpreted as columns. Given a point $(x_1, x_2, \ldots, x_n)$ in R^n we introduce polar coordinates as follows. Set $r^2 = \sum x_i^2$ and, for j satisfying $1 \leq j < n$, define θ_j to be the angle between the j^{th} coordinate vector and the vector $(0,0,\ldots,0,x_j,x_{j+1},\ldots,x_n)$. Thus

$$\theta_j = \arccos\ [x_j/(x_j^2 + \ldots + x_n^2)^{1/2}] \quad \theta_j \in [0,\pi] \quad \text{for} \quad 1 \leq j < n-1\ ,$$

$$\theta_{n-1} = \begin{cases} \arccos\,[x_{n-1}/(x_{n-1}^2 + x_n^2)^{1/2}] & \text{if} \quad x_n \geq 0 \\ 2\pi - \arccos\,[x_{n-1}/(x_{n-1}^2 + x_n^2)^{1/2}] & \text{if} \quad x_n < 0 \end{cases} \qquad \theta_{n-1} \in [0,2\pi)\ .$$

Proceeding inductively, we can show that

$$x_1 = r\cos\theta_1$$
$$x_j = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{j-1}\cos\theta_j \quad \text{for} \quad 1 < j < n\,.$$
$$x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1}$$

The Jacobian of the transform $(r,\theta_1,\theta_2,\ldots,\theta_{n-1}) \to (x_1,x_2,\ldots,x_n)$ is calculated inductively to be

$$r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\ldots(\sin\theta_{n-2}).$$

Thus the volume element in polar coordinates is

$$dV = r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})d\theta_1 d\theta_2\cdots d\theta_{n-1}dr$$

and the surface area element on the unit sphere S is

$$dw = (\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})d\theta_1 d\theta_2\cdots d\theta_{n-1}.$$

We shall have occasion to measure the surface area of a subset of S interior to a ball and the following lemma is useful.

Lemma 1.1.1 For $\eta \in S$ and $\lambda > 0$, set $A = \{x \in S : |x - \eta| < \lambda\}$. Then

$$w(A) = M\int_0^{\mu}(\sin\theta)^{n-2}d\theta$$

where $\mu = \arccos(1 - \lambda^2/2)$, and M is an absolute constant.

Proof. We may as well assume that $\eta = (1,0,0,\ldots,0)$ and then, for $x \in S$, $|\eta - x|^2 = 2(1 - x_1)$ where $x = (x_1,x_2,\ldots,x_n)$. From this it follows that $A = \{x \in S : 1 - \lambda^2/2 \le x_1 \le 1\}$. Recall however that $x_1 = \cos\theta_1$ and the lemma follows when we integrate the surface area element over A and set

$$M = \int_0^{2\pi}\int_0^{\pi}\ldots\int_0^{\pi}(\sin\theta_2)^{n-3}(\sin\theta_3)^{n-4}\cdots(\sin\theta_{n-2})d\theta_2\cdots d\theta_{n-1}$$

which is an absolute constant — namely, the (n-2)-dimensional Lebesgue measure of the unit sphere in R^{n-1}. □

We shall be much concerned in later chapters with the Hausdorff measure and dimension of various subsets of the unit sphere, and we briefly review the definitions. Suppose E is a Borel set in R^n and $\alpha > 0$ is given, if we denote by $\Delta(x,c)$ the ball centered at x of Euclidean radius c, then we define for $\epsilon > 0$

$$\Lambda_\alpha^\epsilon(E) = \inf\{\sum_{j=1}^{\infty} c_j^\alpha \;:\; E \subset \bigcup \Delta(x_j,c_j)\;;\; c_j \le \epsilon\}.$$

This clearly decreases as ϵ increases and the (possibly infinite) limit

$$\Lambda_\alpha(E) = \lim_{\epsilon \to 0}\Lambda_\alpha^\epsilon(E)$$

exists. This quantity is called the α-dimensional Hausdorff measure of E. The Hausdorff dimension, $d(E)$, of a Borel set E is defined by

$$d(E) = \inf\{\alpha : \Lambda_\alpha(E) = 0\} = \sup\{\alpha : \Lambda_\alpha(E) = +\infty\}. \qquad (1.1.1)$$

A consequence of this definition is that if $0 < \Lambda_\alpha(E) < +\infty$ then $d(E) = \alpha$.

1.2 The Hyperbolic Space

The unit ball B of R^n is a model for n-dimensional hyperbolic space and supports a metric ρ derived from the differential

$$d\rho = \frac{2|dx|}{1-|x|^2}.$$

Lines in the space are arcs of circles orthogonal to the unit sphere S and are geodesics for the metric ρ. Angle is Euclidean angle.

An alternative model of n-dimensional hyperbolic space is the upper half space H of R^n

$$H = \{x = (x_1, x_2, \ldots, x_n) : x_n > 0\}$$

together with the metric ρ derived from the differential

$$d\rho = \frac{|dx|}{x_n}.$$

Lines in this space are arcs of circles orthogonal to the plane $\{x : x_n = 0\}$ and are geodesics for the metric ρ.

Note that we use ρ for the metric in both the ball and the upper half space model — no confusion should arise. We shall be working almost entirely in the ball but will from time to time be using the upper half space model, each has its own particular advantages.

There is a wealth of information on hyperbolic geometry to be found in [Beardon, 1983] and in [Ahlfors, 1981, Chapter 3] — we will quote extensively from these sources. The formula for the hyperbolic distance between a point and a line is to be found in [Beardon, 1983, p.162] but not in the form best suited for our purposes. We state the result now, although we will not be in a position to prove it until the next section.

Theorem 1.2.1 Suppose $a \in B$ and $\eta, \xi \in S$, $\eta \neq \xi$. Let s be the hyperbolic distance from a to the geodesic joining ξ and η then

$$\cosh s = \frac{2|a-\xi||a-\eta|}{|\xi-\eta|(1-|a|^2)}.$$

For later purposes we introduce the notion of "shadows". Suppose $a \in B$ and

$\delta > 0$ are given, and let $\Delta(a,\delta)$ denote the hyperbolic ball of center a and radius δ. The set $b(0:a,\delta)$ is the projection from the origin onto S of the ball $\Delta(a,\delta)$ and so every point of $b(0:a,\delta)$ lies in the shadow of the ball. A point $\xi \in S$ clearly belongs to $b(0:a,\delta)$ if and only if the radius to ξ passes within a hyperbolic distance δ of a. From theorem 1.2.1 this is equivalent to

$$|a-\xi||a+\xi| < (1-|a|^2)\cosh\delta \quad \text{and} \quad |a+\xi| > |a-\xi|. \qquad (1.2.1)$$

It will be useful to compute the w-measure of the shadow $b(0:a,\delta)$ and we have the following.

Theorem 1.2.2 For $\delta > 0$

$$w(b(0:a,\delta)) \sim \frac{M\,(\cosh^2\delta - 1)^{(n-1)/2}}{n-1}(1-|a|)^{n-1}$$

uniformly as $|a| \to 1$ in B, where M is the constant of lemma 1.1.1.

Proof. Suppose $a \in B$ and $\xi \in S$ then

$$|a-\xi|^2 = 1 + |a|^2 - 2a.\xi,$$
$$|a+\xi|^2 = 1 + |a|^2 + 2a.\xi,$$

and

$$\left|\xi - \frac{a}{|a|}\right|^2 = 2 - \frac{2a.\xi}{|a|}.$$

We use these equations and inequality (1.2.1) to note that $\xi \in b(0:a,\delta)$ if and only if

$$(1+|a|^2)^2 - 4(a.\xi)^2 < (1-|a|^2)^2\cosh^2\delta$$

in other words,

$$(1+|a|^2)^2 - (2 - |\xi - a/|a||^2)^2\,|a|^2 < (1-|a|^2)^2\cosh^2\delta.$$

This reduces to

$$\left|\xi - \frac{a}{|a|}\right| < \frac{[2|a| - ((1+|a|^2)^2 - (1-|a|^2)^2\cosh^2\delta)^{1/2}]^{1/2}}{|a|^{1/2}} = \lambda,$$

say. A routine calculation shows that

$$\lambda \sim (1-|a|)(\cosh^2\delta - 1)^{1/2} \quad as \quad |a| \to 1. \qquad (1.2.2)$$

Using lemma 1.1.1 we see that

$$w(S(a,\delta)) = M \int_0^{\mu} (\sin\ \theta)^{n-2} d\theta \tag{1.2.3}$$

where $\mu = \arccos\ (1 - \lambda^2/2)$. Clearly $\mu \sim (1 - |a|)(\cosh^2\delta - 1)^{1/2}$ as $|a| \to 1$ (from (1.2.2)) and, for θ between 0 and μ, we approximate $\sin\theta$ by θ. The theorem now follows from (1.2.3). □

In order to measure the size of various subsets of the unit sphere, the following definition is useful. If $a \in B$ and $k, \alpha > 0$ we define

$$I(a:k,\alpha) = \{x \in S : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^{\alpha}\}. \tag{1.2.4}$$

We next consider cones at a point $\xi \in S$. If $x \in B$, $\xi \in S$ and λ satisfies $0 < \lambda < \pi/2$ then we say x belongs to the cone at ξ of opening λ if the angle between the vectors ξ and $\xi - x$ is at most λ and, further, $|x - \xi| < 2\cos\lambda$. The cosine of the angle between ξ and $\xi - x$ is calculated to be

$$\frac{\xi.(\xi - x)}{|\xi||\xi - x|} = \frac{2 - 2\xi.x}{2|\xi - x|} = \frac{(\xi - x).(\xi - x) + 1 - |x|^2}{2|\xi - x|} = \frac{|\xi - x|^2 + 1 - |x|^2}{2|\xi - x|}$$

and we have proved the following.

Lemma 1.2.3 If $x \in B$, $\xi \in S$ and λ satisfies $0 < \lambda < \pi/2$ then x belongs to the cone at ξ of opening λ if and only if $|x - \xi| < 2\cos\lambda$ and

$$\frac{|\xi - x|^2 + 1 - |x|^2}{2|\xi - x|} > \cos\lambda.$$

Theorem 1.2.4 Suppose $\xi \in S$ and $\{x_n\}$ is a sequence of points of B with $|x_n| \to 1$ as $n \to \infty$. The following are equivalent.

1. There exists $a > 0$ such that, for n large enough, x_n lies in the cone of opening a at ξ.
2. There exists $b > 1$ such that, for n large enough,
$$|x_n - \xi| < b(1 - |x_n|).$$
3. There exists $c > 0$ such that, for n large enough,
$$\xi \in I(x_n : c, 1).$$
4. There exists $d > 0$ such that if l is any geodesic ending at ξ then, for n large enough, $\rho(x_n, l) < d$.

Proof. If (1) is true we note that, for n large enough,

$$\frac{|\xi - x_n|^2 + 1 - |x_n|^2}{2|\xi - x_n|} > \cos a$$

from lemma 1.2.3. Since $|x_n| \to 1$ we see that, given $\epsilon > 0$, for n large enough,

$$|\xi - x_n| < \frac{(1 - |x_n|)}{(\cos a - \epsilon)}.$$

Thus (1) implies (2). Now suppose (2) is true and we note that

$$|\xi - \frac{x_n}{|x_n|}|^2 = \frac{|\xi - x_n|^2 - (1 - |x_n|)^2}{|x_n|}.$$

Therefore

$$|\xi - \frac{x_n}{|x_n|}| < \frac{(b^2 - 1)^{1/2}(1 - |x_n|)}{|x_n|^{1/2}}.$$

We may take $c = (b^2 - 1)^{1/2} + \epsilon$ and (3) follows. Assuming (3) we let l be a geodesic ending at ξ and suppose that η is the other end point of l. From theorem 1.2.1 we see that

$$\cosh \rho(x_n, l) = \frac{2|x_n - \xi||x_n - \eta|}{|\xi - \eta|(1 - |x_n|^2)}$$

which is asymptotic (as $n \to \infty$) to $\dfrac{|x_n - \xi|}{(1 - |x_n|)}$. However, from (3)

$$\frac{|\xi - x_n|}{1 - |x_n|} < (1 + c^2)^{1/2}$$

for n large enough and (4) is true. Finally, we suppose (4) is true and note from theorem 1.2.1 that, for n large enough,

$$\frac{2|x_n - \xi||x_n - \eta|}{|\xi - \eta|(1 - |x_n|^2)} < \cosh d.$$

Thus, from our remarks above, if $\epsilon > 0$ is given then $\dfrac{|x_n - \xi|}{(1 - |x_n|)} < \cosh d + \epsilon$.

For n large enough,

$$\frac{|\xi - x_n|^2 + 1 - |x_n|^2}{2|\xi - x_n|} > \frac{1}{\cosh d + \epsilon} - \epsilon$$

and (1) follows from lemma 1.2.3. This completes the proof of the theorem. □

It should be noted, from our working above, that the constants a, b, c, d are related by

$$b \approx \frac{1}{\cos a} \approx \cosh d \approx (1+c^2)^{1/2}$$

In other words, (1) implies (2) with $b = \frac{1}{\cos a} + \epsilon$ for any $\epsilon > 0$ and (2) implies (1) with $a = \arccos(1/b) + \epsilon$ for any $\epsilon > 0$. Similar remarks hold for the relations between b, c and d.

We next consider horospheres. A **horosphere** at $\xi \in S$ is a sphere in R^n which is internally tangent to the unit sphere S at ξ. A **horoball** is the interior of a horosphere.

Theorem 1.2.5 If $\xi \in S$, $x \in B$ and $0 < k < 1$ then x is on the horosphere at ξ of Euclidean radius k if and only if

$$(1-|x|^2)|x-\xi|^{-2} = \frac{1-k}{k}.$$

The point x is in the horoball at ξ of radius k if and only if

$$(1-|x|^2)|x-\xi|^{-2} > \frac{1-k}{k}.$$

Proof. Suppose $x \in B$ with $(1-|x|^2)|x-\xi|^{-2} = \frac{1-k}{k}$ then

$$1-|x|^2 = \frac{1-k}{k}(x-\xi).(x-\xi) = \frac{1-k}{k}(|x|^2+1-2x.\xi)$$

and so

$$2x.\xi = \frac{(|x|^2+1-2k)}{(1-k)}. \tag{1.2.5}$$

Now the square of the distance of x from the center of the horosphere is

$$\begin{aligned} |x-(1-k)\xi|^2 &= |x|^2 + (1-k)^2 - 2(1-k)x.\xi \\ &= |x|^2 + (1-k)^2 - (|x|^2+1-2k) \\ &= k^2 \end{aligned}$$

where we substituted for $x.\xi$ from (1.2.5) above. Thus x lies on the horosphere. Our argument is clearly reversible and we have the if and only if condition. The statement concerning the horoball is an easy modification. □

1.3 Moebius Transforms

In this section we consider Moebius transforms acting in R^n, derive several of their properties, and show that those preserving the ball are isometries of hyperbolic space. For a full and detailed account, the reader is referred to [Beardon, 1983, Chapter 3] or [Ahlfors, 1981, Chapter 2]. Much of what we do in this section is in fact a slightly compressed version of Ahlfors' account. We start by defining a **similarity** as a map $R^n \rightarrow R^n$ given by

$$x \longrightarrow m\,x + b$$

where $b \in R^n$ and m is a **conformal** matrix (i.e., a positive constant multiple of an orthogonal matrix). Reflection in the unit sphere is given by

$$x \longrightarrow x^* = J(x) = \frac{x}{|x|^2}$$

and we define the full Moebius group as the group generated by J and all the similarities.

The derivative of a self map of R^n is the Jacobian matrix, and we will use the prime notation. Observe that the derivative of a similarity $\gamma(x) = m\,x + b$ is the constant matrix m. In order to write down the derivative of J we introduce the matrix $Q(x)$ by

$$Q(x)_{i\,j} = \frac{x_i\,x_j}{|x|^2}$$

and leave it as an exercise to check that, for $x \neq 0$,

$$J'(x) = \frac{1}{|x|^2}[I - 2Q(x)]\,. \tag{1.3.1}$$

Since $Q^2 = Q$ we have $[I - 2Q]^2 = I$ and it follows that $I - 2Q$ is an orthogonal matrix. For each $x \neq 0$, $J'(x)$ is a conformal matrix.

Use of the chain rule shows that $\gamma'(x)$ is a conformal matrix for any Moebius γ — in other words, Moebius transforms are **conformal.** For any Moebius γ we denote by $|\gamma'(x)|$ the positive number such that $\gamma'(x)/|\gamma'(x)|$ is orthogonal. Thus $|\gamma'(x)|$ is the linear change of scale at x, the same in all directions.

The following equation, which will be fundamental to our work, follows from the chain rule and application of (1.3.1).

$$|\gamma(x) - \gamma(y)| = |\gamma'(x)|^{1/2}\,|\gamma'(y)|^{1/2}\,|x - y|\,. \tag{1.3.2}$$

Application of (1.3.2) proves the invariance of the absolute cross ratio

$$|a, b, c, d| = \frac{|a - c|}{|a - d|} \cdot \frac{|b - d|}{|b - c|}$$

in the sense that

$$|\gamma(a), \gamma(b), \gamma(c), \gamma(d)| = |a, b, c, d|. \tag{1.3.3}$$

We denote by $GM(B)$ the subgroup of the full Moebius group which leaves the unit ball invariant, and prove the following lemma.

Lemma 1.3.1 If $\gamma \in GM(B)$ and $\gamma(0) = 0$ then γ is a rotation. In other words, $\gamma(x) = k\,x$ where k is an orthogonal matrix.

Proof. (Ahlfors, 1981 p.21) Let us suppose first that $\gamma(\infty) = \infty$. Since

$$|\gamma(x), \gamma(y), 0, \infty| = |x, y, 0, \infty|$$

we deduce that $|\gamma(x)|/|x| = \lambda$, a constant. The equation

$$|\gamma(x), 0, \gamma(y), \infty| = |x, 0, y, \infty|$$

yields $|\gamma(x) - \gamma(y)|^2 = \lambda^2 |x - y|^2$. It follows that for any x $\gamma(x) = \lambda^2 x$ and so

$$|\gamma(x + y) - \gamma(x) - \gamma(y)|^2 = \lambda^2 |(x + y) - x - y|^2 = 0.$$

Thus $\gamma(x + y) = \gamma(x) + \gamma(y)$. From this we deduce that γ' is constant, and $\gamma(x) = m\,x$ with a constant conformal matrix m. Since $|mx| = 1$ for $|x| = 1$ we have that m is orthogonal as required.

If we now suppose that $\gamma^{-1}(\infty) = b \neq \infty$ then the Moebius transform $\gamma((x - b^*)^* + b)$ fixes 0 and ∞ and so, by our working above, we have

$$\gamma(x) = m\,((x - b)^* + b^*)$$

for a constant conformal matrix m. But γ preserves the unit ball and so $|(x - b)^* + b^*|$ is a constant for $|x| = 1$. But

$$|(x - b)^* + b^*| = \frac{|x|}{|x - b||b|}$$

and so $|x - b|$ is constant on the unit sphere — this is impossible since $b \neq 0$, and the contradiction completes the proof of the lemma. □

With this result in hand, we determine the form of the most general Moebius transform preserving B. From now on, indeed for the rest of the book, we confine attention to the orientation preserving Moebius transforms preserving B. These are the transforms containing an even number of factors J and sense preserving similarities. Thus, from this point on, $M(B)$ will be used to denote the group of orientation preserving Moebius transforms preserving B.

Given $a \in B$, $a \neq 0$, let S_a be the sphere centered at a^* and of radius $(1 - |a|^2)^{1/2}/|a|$ — this sphere intersects the unit sphere orthogonally. Let σ_a denote reflection in S_a so that

$$\sigma_a(x) = a^* + (|a^*|^2 - 1)(x - a^*)^*.$$

We write R_a for the reflection in the plane through the origin perpendicular to a and define the canonical map

$$T_a(x) = R_a \circ \sigma_a(x).$$

As an immediate consequence of lemma 1.3.1, we note that the most general Moebius transform preserving the ball and mapping a to 0 can be written as T_a followed by a rotation.

We need to derive explicit formulae for $T_a(x)$ and note first that the reflection R_a amounts to multiplication by the matrix $I - 2Q(a)$. This leads immediately to

$$T_a(x) = -a^* + (|a^*|^2 - 1)[I - 2Q(a)](x - a^*)^*. \tag{1.3.4}$$

To proceed further we need a lemma.

Lemma 1.3.2 For any x, y

$$(x - y^*)^* + y = |y|^2[I - 2Q(y)](x^* - y)^*.$$

Proof. (Ahlfors, 1981 p.22) Consider two transforms

$$A(x) = (x - y^*)^* \quad \text{and} \quad B(x) = |y|^2[I - 2Q(y)](x^* - y)^* - y$$

and note that $0, \infty$ are fixed points of AB^{-1}. As in the proof of lemma 1.3.1, we see that $(AB^{-1})'$ is constant. Now note that

$$(AB^{-1})'(x) = A'(B^{-1}(x)).(B'(B^{-1}(x)))^{-1} = (A'(B')^{-1}) \circ B^{-1}(x)$$

which is constant. Thus, for some λ, $A'(x) = \lambda B'(x)$. Differentiation yields

$$A'(x) = \frac{I - 2Q(x - y^*)}{|x - y^*|^2}$$

$$B'(x) = \frac{|y|^2(I - 2Q(y))(I - 2Q(x^* - y))(I - 2Q(x))}{|x^* - y|^2\,|x|^2}.$$

So for $x = y$ we obtain

$$A'(y) = B'(y) = \frac{(I - 2Q(y))\,|y|^2}{(1 - |y|^2)^2}.$$

It follows that $A'(x) = B'(x)$ for all x and, since $A(0) = B(0) = -y$, we have $A(x) = B(x)$ and this completes the proof. □

Note from the formulae for A' and B' that