

FUNDAMENTAL SOLUTIONS IN ELASTODYNAMICS

This work is a compilation of fundamental solutions (or Green's functions) for classical or canonical problems in elastodynamics, presented with a common format and notation. These formulas describe the displacements and stresses elicited by transient and harmonic sources in solid elastic media such as full spaces, half-spaces, and strata and plates in both two and three dimensions, using the three major coordinate systems (Cartesian, cylindrical, and spherical). Such formulas are useful for numerical methods and practical application to problems of wave propagation in elasticity, soil dynamics, earthquake engineering, mechanical vibration, or geophysics. Together with the plots of the response functions, this work should serve as a valuable reference to engineers and scientists alike. These formulas were heretofore found only scattered throughout the literature. The solutions are tabulated without proof, but giving reference to appropriate modern papers and books containing full derivations. Most formulas in the book have been programmed and tested within the MATLAB environment, and the programs thus developed are both listed and available for free download.

Eduardo Kausel earned his first professional degree in 1967, graduating as a civil engineer from the University of Chile, and then worked at Chile's National Electricity Company. In 1969 he carried out postgraduate studies at the Technical University in Darmstadt. He earned his Master of Science (1972) and Doctor of Science (1974) degrees from MIT. Following graduation, Dr. Kausel worked at Stone and Webster Engineering Corporation in Boston, and then joined the MIT faculty in 1978, where he has remained since. He is a registered professional engineer in the State of Massachusetts, is a senior member of various professional organizations (ASCE, SSA, EERI, IACMG), and has extensive experience as a consulting engineer.

Among the honors he has received are a 1989 Japanese Government Research Award for Foreign Specialists from the Science and Technology Agency, a 1992 Honorary Faculty Membership in Epsilon Chi, the 1994 Konrad Zuse Guest Professorship at the University of Hamburg in Germany, the Humboldt Prize from the German Government in 2000, and the 2001 MIT-CEE Award for Conspicuously Effective Teaching.

Dr. Kausel is best known for his work on dynamic soil–structure interaction, and for his very successful Green's functions (fundamental solutions) for the dynamic analysis of layered media, which are incorporated in a now widely used program. Dr. Kausel is the author of more than 150 technical papers and reports in the areas of structural dynamics, earthquake engineering, and computational mechanics.

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Fundamental Solutions in Elastodynamics

A Compendium

EDUARDO KAUSEL

Massachusetts Institute of Technology



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Preface

We present in this work a collection of fundamental solutions, or so-called *Green's functions*, for some classical or canonical problems in elastodynamics. Such formulas provide the dynamic response functions for transient point sources acting within isotropic, elastic media, in both the frequency domain and the time domain, and in both two and three dimensions. The bodies considered are full spaces, half-spaces, and plates of infinite lateral extent, while the sources range from point and line forces to torques, seismic moments, and pressure pulses. By appropriate convolutions, these solutions can be extended to spatially distributed sources and/or sources with an arbitrary variation in time.

These fundamental solutions, as their name implies, constitute invaluable tools for a large class of numerical solution techniques for wave propagation problems in elasticity, soil dynamics, earthquake engineering, or geophysics. Examples are the Boundary Integral (or element) Method (BIM), which is often used to obtain the solution to wave propagation problems in finite bodies of irregular shape, even while working with the Green's functions for a full space.

The solutions included herein are found scattered throughout the literature, and no single book was found to deal with them all in one place. In addition, each author, paper, or book uses sign conventions and symbols that differ from one another, or they include only partial results, say only the solution in the frequency domain or for some particular value of Poisson's ratio. Sometimes, published results are also displayed in unconventional manners, for example, taking forces to be positive down, but displacements up, or scaling the displays in unusual ways or using too small a scale, and so forth. Thus, it was felt that a compendium of the known solutions in a common format would serve a useful purpose. With this in mind, we use throughout a consistent notation, coordinate systems, and sign convention, which should greatly facilitate the application of these fundamental solutions. Also, while we anguished initially at the choice of symbols for the angles in spherical coordinates, we decided in the end to use θ for the azimuth and ϕ for the polar angle. Although this contravenes the common notation, it provides consistency between spherical and cylindrical coordinates and eases the transition between one and the other system.

We tabulate these solutions herein without proof, giving reference to appropriate modern papers and books containing full derivations, but making no effort at establishing the original sources of the derivations or, for that matter, providing a historical account of

these solutions. In some cases, we give no references, in which case we have developed the formulae ourselves using established methods, either because an appropriate reference was not known to us, not readily available, or for purely pragmatic reasons. Yet, recognizing that these are all classical problems, we do not claim to have discovered new formulas. Also, the tables may not necessarily be complete in that solutions for some additional classical problems, or important extensions to these, may exist of which we may be unaware. If and when these are brought to our attention, we shall be happy to consider them with proper credit when preparing a revised version of this work.

Finally, we have programmed most formulas within the MATLAB or other programming environment, and provide plots of response functions that could be used to verify the correctness of a particular implementation. Also, we have made every effort at checking the formulas themselves for correctness and dimensional consistency. Nonetheless, the possibility always exists that errors may remain undetected in some of these formulas. If the reader should find any such errors, we shall be thankful if they are brought to our attention.

Eduardo Kausel

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