The investigation of difficult things

Essays on Newton and the history of the exact sciences
in honour of D. T. Whiteside

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I Lunar velocity in the Ptolemaic tradition

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Introduction

In ancient and medieval astronomy lunar velocity is used to compute the duration of eclipses and the time from mean syzygy to true syzygy.[1] In Ptolemy's *Almagest*, a procedure is given for computing 'instantaneous' velocity without justification, based on the simple lunar model. A different rule is given by Regiomontanus (d. 1476), also without justification, that seems to reflect a recognition of the effect of the second lunar inequality, known as the evection, on the 'instantaneous' lunar velocity at syzygy. It is perhaps curious that, although instantaneous velocity could not be defined in this period, some astronomers could compute it correctly (i.e., according to the models for lunar motions they held to be true). Indeed, even the concept of uniform velocity was problematic.

Thus, for example, in Archimedes' *On Spirals* uniform velocity is described in terms of the proportionality of ratios formed between like quantities (e.g., length to length, or time to time). This appears in the first two propositions: Prop. (1) 'If a point moves at a uniform rate along a line, and two lengths be taken on it, they will be proportional to the times describing it'; and Prop. (2) 'If each of two points on different lines respectively move along them at a uniform rate, and if lengths be taken, one on each line, forming pairs, such that each pair are described in equal times, the lengths will be proportional'.[2] In other words, a ratio of length to time (or an angle to time) was not considered legitimate in ancient and medieval mathematics.[3]

1 The term 'syzygy' is used for conjunction, or opposition, of the Sun and the Moon in celestial longitude.

2 T. L. Heath (trans.), *The Works of Archimedes*, Cambridge University Press, 1897 (Dover reprint: n.d.), p. 155.

3 Cf. E. J. Dijksterhuis, *The Mechanization of the World Picture*, Oxford University Press, 1961, pp. 191–2.

It is possible that Regiomontanus' rule was used to compute lunar velocity tables before he stated it; yet, no earlier source has been found in which this procedure is discussed.[4] In this paper we shall consider the rule for computing lunar velocity according to Ptolemy (*fl. ca.* 150) based on his simple lunar model (note that he gives no table of lunar velocity),[5] the table for lunar velocity by al-Battānī (d. 929) computed from Ptolemy's simple lunar model,[6] the two tables for lunar velocity by Levi ben Gerson (d. 1344),[7] and some of the tables for lunar velocity in the Alfonsine corpus (composed in the fourteenth century),[8] as well as the rule stated by Regiomontanus.[9]

4 Dr J. L. Mancha informs me that in one of the manuscripts with a lunar velocity table ascribed to John of Genoa, we find 'Hic sciendum quod in tabula motus lune est computatum illud quod contingit lune propter equationem centri' (Paris, Bibliothèque Nationale, abbr. BN, lat. 7282, 129v), that is, 'One should understand here that what has been computed in this table for the motion of the Moon is that which affects the Moon due to the equation of center'. A similar remark appears in a manuscript containing John of Montfort's version of this lunar velocity table: Paris, BN lat. 7283, 44v. The expression 'equation of center' for the Moon in the Alfonsine tables refers to the entries in column 2 of the lunar correction table (whose argument is the double elongation of the Moon from the Sun) that does not have any role in Ptolemy's simple lunar model. These passages suggest that those who computed these lunar velocity tables knew what they were doing, but they do not reveal the mathematical argument that led to the method of calculation. For a discussion of two alternative explanations for the method of calculation, see Section IV.

5 Ptolemy's simple lunar model is described in the *Almagest* IV. For a discussion of Ptolemy's lunar models see O. Neugebauer, *The Exact Sciences in Antiquity* (Providence: Brown University Press, 1957), pp. 192–9; and O. Pedersen, *A Survey of the Almagest* (Odense: University Press, 1974), pp. 159–202.

6 Al-Battānī's table of lunar velocity appears in C. A. Nallino, *Al-Battānī sive Albatenii Opus Astronomicum.* 3 vols. (Milan 1899–1907), vol. 2, p. 88. For a discussion of the way in which it was computed, see B. R. Goldstein, *The Astronomical Tables of Levi ben Gerson* (New Haven: Connecticut Academy of Arts and Sciences, 1974), pp. 108–13. For the presence of this table among the Toledan tables, see G. J. Toomer, 'A survey of the Toledan tables', *Osiris,* 15 (1968), pp. 5–174, on p. 86.

7 See Goldstein, *Astronomical Tables*, Table 22; see also B. R. Goldstein, *The Astronomy of Levi ben Gerson (1288–1344)*, Studies in the History of the History of Mathematics and the Physical Sciences, 11 (New York, Berlin: Springer-Verlag, 1985).

8 See E. Poulle, *Les Tables Alphonsines* (Paris: Editions du CNRS, 1984), pp. 210–13; cf. E. Poulle, 'The Alfonsine tables and Alfonso X of Castille', *Journal for the History of Astronomy*, 19 (1988), pp. 97–113. There are three versions of a table for lunar velocity that differ from al-Battānī's table, and they are ascribed to John of Lignères (*fl. ca.* 1330), John of Genoa, and John of Montfort. See E. Poulle, 'John of Lignères', *Dictionary of Scientific Biography*, 7 (1973), pp. 122–8. Little is known about John of Genoa and John of Montfort, although it is clear that they are somehow associated with the Parisian astronomers in the 1330s who compiled the Latin version of the Alfonsine tables (see P. Duhem, *Le Système du monde* [Paris: Hermann, 1954],

I

Ptolemy's procedure for computing the lunar velocity at syzygy (in degrees per hour) for a given lunar anomaly, α, expressed algebraically, is the following:[10]

$$v(\alpha) = 0;32,56 + 0;32,40 \cdot \Delta \tag{1}$$

where $0;32,56^\circ/^h$ is the hourly mean motion of the Moon in longitude ($\bar{v}$), $0;32,40^\circ/^h$ is the hourly mean motion in anomaly ($\bar{v}_\alpha$), and Δ is the difference in the correction for anomaly, c, in the simple model from α to $\alpha + 1^\circ$:

$$\Delta = c(\alpha + 1^\circ) - c(\alpha) \tag{2}$$

where α is an integer.[11] This yields reasonably good agreement with al-Battānī's table for lunar velocity.[12] Toomer comments that, since (as Ptolemy himself states) there is no effect of Ptolemy's second lunar inequality on the lunar position at syzygy, the second inequality can be ignored in computing the velocity there; but this statement is clearly wrong.[13] When the value of a trigonometric function is 0, its derivative is often 1: hence, we should expect the maximum effect on velocity when the second lunar inequality is 0°. Pedersen demonstrated (using partial derivatives: see Appendix 1) that the second lunar inequality may not be ignored and showed that of the three terms that define the lunar velocity according to Ptolemy's complete lunar model [see eq. (a1)], the first is identical to the first term in eq. (1), the second vanishes at syzygy, and the third term corresponds to the second term in eq. (1). However, in the second term in eq. (1) Ptolemy used the wrong value for the mean anomaly by disregarding the effect of the second inequality. As we shall see, Regiomontanus computed what Pedersen proved for the second term in eq.

vol. 4, pp. 74–5). As Dr Mancha informs me, John of Montfort's tables are dated January 1332 (MS Paris, BN lat. 7283, 44r) and, surprisingly, John of Genoa's tables are also dated 1332 in the same MS (Paris, BN lat. 7283, 45r). For the presence of al-Battānī's table of lunar velocity in the Alfonsine corpus, see Poulle, *Les Tables Alphonsines*, p. 210. A variant on this table is also found in the Alfonsine corpus: see B. R. Goldstein, 'Solar and lunar velocities in the Alfonsine tables', *Historia Mathematica*, 7 (1980), pp. 134–40.

9 Regiomontanus, *Epitome of the Almagest*: *Epytoma Joannis de Monte Regio in Almagestum Ptolomei* (Venice: J. Hamman-Hertzog, 1496); reprinted in Regiomontanus, *Opera Collectanea*, ed. F. Schmeidler (Osnabrück 1972). The rule for computing lunar velocity occurs in VI, 4.

10 Almagest VI,4 (G. J. Toomer [trans.], *Ptolemy's Almagest*, Berlin, New York: Springer-Verlag, 1984, p. 282).

11 Note that the values for $c(\alpha)$ are tabulated at degree intervals in al-Battānī's tables, ed. Nallino, 1907, vol. 2, pp. 78–83, col. 3.

12 Ed. Nallino, *al-Battānī*, vol. 2, p. 88.

13 See Toomer, *Almagest*, p. 282 n15 where he cites the discussion in Pedersen 1974, p. 226.

(1), above, and we suggest that Regiomontanus (or his predecessor) formulated this rule for computing lunar velocity at syzygy based on his understanding of Ptolemy's procedure.

To justify Ptolemy's rule (i.e., ignoring the second lunar inequality), let us consider λ, the lunar longitude, and $\bar{\lambda}$, the mean lunar longitude, where:

$$\lambda = \bar{\lambda} + c. \tag{3}$$

For λ_1 and λ_2 this means that

$$\lambda_1 = \bar{\lambda}_1 + c_1$$

and

$$\lambda_2 = \bar{\lambda}_2 + c_2.$$

Thus,

$$\Delta\lambda = \lambda_2 - \lambda_1 = \Delta\bar{\lambda} + \Delta c$$

and

$$v = \frac{\Delta\lambda}{\Delta t} = \frac{\Delta\bar{\lambda}}{\Delta t} + \frac{\Delta c}{\Delta t}. \tag{4}$$

Now

$$\Delta\alpha = \bar{v}_\alpha \cdot \Delta t$$

or

$$\Delta t = \Delta\alpha / \bar{v}_\alpha,$$

and

$$\frac{\Delta\bar{\lambda}}{\Delta t} = \bar{v}.$$

Hence, eq. (4) becomes

$$v = \bar{v} + \frac{\Delta c}{\Delta\alpha / \bar{v}_\alpha}$$

or

$$v = \bar{v} + \bar{v}_\alpha \cdot \Delta c, \tag{5}$$

where

$$\Delta\alpha = 1° \tag{6}$$

as in eq. (2), above. Equation (5) corresponds to Ptolemy's instructions for computing the lunar velocity (see eq. (1), above: $\Delta c = \Delta$ when $\Delta\alpha = 1°$); hence, given his assumption that the second inequality can be ignored, his formula is correct. Note that eq. (5) can also be obtained from eq. (4) by using the relation:

$$\frac{\Delta c}{\Delta t} = \frac{\Delta c}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\Delta t}$$

which, with the condition in eq. (6), leads directly to eq. (5) because

$$\bar{v}_\alpha = \frac{\Delta\alpha}{\Delta t}.$$

Table 1. *Lunar velocity with al-Battānī's lunar corrections*

I argument (°)	II Δ (°)	III v: comp. (°/h)	IV v: text (°/h)	V v: comp. (°/h)
0	−0; 4,50	0;30,18	0;30,18	0;29,34
30	−0; 4,18	0;30,36	0;30,35	0;29,56
60	−0; 2,47	0;31,25	0;31,25	0;31, 0
90	−0; 0,24	0;32,43	0;32,41	0;32,39
120	0; 2,24	0;34,14	0;34,14	0;34,36
150	0; 4,47	0;35,32	0;35,31	0;36,16
180	0; 5,45	0;36, 4	0;36, 4	0;36,56

The entries in cols. II and IV have been derived from al-Battānī's tables. The entries in col III have been computed from the entries in col. II according to Ptolemy's rule, whereas the entries in col. V have been computed from the entries in col. II according to Regiomontanus' rule.

Thus, in eq. (4)

$$v = \frac{\Delta\bar{\lambda}}{\Delta t} + \frac{\Delta c}{\Delta t}$$

or

$$v = \bar{v} + \frac{\Delta c}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\Delta t}$$

$$= \bar{v} + \bar{v}_\alpha \cdot \Delta c,$$

where

$$\Delta\alpha = 1°.$$

In Table 1, col. II, we display al-Battānī's values for Δ; in col. III the lunar velocities computed from them according to eq. (1); and in col. IV the entries in al-Battānī's table for lunar velocity. The agreement is very good [i.e., T(ext) − C(omputation) = 0″ or −1″] except for $\alpha = 90°$ (where T − C = −2″). In order to get the computed velocity equal to the velocity in the text for 90°, the value for Δ would have to be −0;0,28° instead of −0;0,24°, but with −0;0,25° the velocity would be 0;32,42 which is only 1″ greater than the value in the text (this would require that $c(91°) = -5;0,27°$ instead of −5;0,26°, the value that appears in al-Battānī's table). Another method proposed for computing these values (see Appendix 2) yields exact agreement for $\alpha = 90°$ (as well as for $\alpha = 0°$ and $\alpha = 180°$) and nowhere do the differences between text and computation exceed 1″. Since al-Battānī does not say how the entries in his table were computed, we cannot determine his method with certainty. But, if we take agreement with recomputation as a guide, it seems slightly more likely that he used the method described in Appendix 2 according to which the progress of the Moon is computed for an interval of an hour.

II

According to Regiomontanus' *Epitome of the Almagest* VI,4, the lunar velocity at syzygy may be computed from the following formula:

$$v(\alpha') = 0;32,56 + 0;41,49 \cdot \Delta. \tag{7}$$

This formula differs from eq. (1) in significant ways. Firstly, the values $0;32,56^{\circ/h}$ and $0;41,49^{\circ/h}$ represent the Moon's hourly mean motion in longitude ($\bar{v}$), and the corrected hourly mean motion in anomaly at syzygy (v_α), respectively. Secondly, the argument is the true lunar anomaly, α', that appears in Ptolemy's complete lunar model, rather than the mean anomaly, α. The expression, Δ, is defined as before in eq. (2). We can express eq. (7) more generally as

$$v(\alpha') = \bar{v} + v_\alpha \cdot \Delta c \tag{8}$$

which has the advantage that we can substitute other values for the basic parameters.

It has been suggested that v_α, the corrected hourly mean motion in anomaly at syzygy in Regiomontanus' formula, was computed as follows:[14]

$$v_\alpha = 0;32,40 + 1;1 \cdot 0;9 = 0;41,49 \tag{9}$$

where $0;32,40^{\circ/h}$ is the hourly mean motion in anomaly ($\bar{v}_\alpha$), $1;1^{\circ/h}$ is the hourly mean motion in double elongation ($\bar{v}_{2\eta}$), and $0;9^\circ$ is the difference between c' (the correction to the mean anomaly found in column 4 of al-Battānī's lunar correction tables) for arguments of double elongation 0° and 1°. Again, we can express (9) more generally as

$$v_\alpha = \bar{v}_\alpha + \bar{v}_{2\eta} \cdot \Delta c' \tag{10}$$

where $\Delta c'$ is restricted as stated above.

The parameter derived in (9) can be justified by a procedure similar to the one we invoked to justify eq. (1). We define

$$\alpha' = \alpha + c'.$$

14 See N. M. Swerdlow and O. Neugebauer, *Mathematical Astronomy in Copernicus's De Revolutionibus*, Studies in the History of Mathematics and Physical Sciences, 10 (New York, Berlin: Springer-Verlag, 1984), pp. 274–6.

Thus,

$$\Delta\alpha' = \Delta\alpha + \Delta c'$$

and

$$v_\alpha = \bar{v}_\alpha + \frac{\Delta c'}{\Delta t}.$$

Now

$$\bar{v}_{2\eta} = \Delta(2\eta)/\Delta t$$

or

$$\Delta t = \Delta(2\eta)/\bar{v}_{2\eta}.$$

Hence,

$$v_\alpha = \bar{v}_\alpha + \bar{v}_{2\eta} \cdot \Delta c' \tag{11}$$

where

$$\Delta(2\eta) = 1°. \tag{12}$$

Let us compute the hourly lunar velocity (as suggested by Regiomontanus) by the formula in eq. (7). Using the table for corrections due to the lunar anomaly for Ptolemy's simple model in al-Battānī's tables, we compute the velocities displayed in Table 1, col. V, that are certainly different from the velocities based on eq. (1) displayed in Table 1, col. III.

III

We will now investigate the evidence for the use of Ptolemy's complete lunar model for computing lunar velocity in the fourteenth century, more than 100 years before Regiomontanus wrote his *Epitome*, restricting our attention to the two lunar velocity tables in the astronomical works of Levi ben Gerson and some of the lunar velocity tables in the corpus of tables associated with the name of Alfonso, king of Castile (reigned: 1252–84).

It has been argued that one of Levi's lunar velocity tables was computed in the same way as al-Battānī's lunar velocity table except that Levi used his own lunar model at syzygy rather than Ptolemy's model.[15] As a result, there are some small differences between their respective tables. Levi's other lunar velocity table is more puzzling. There seems to be no indication in the text of his *Astronomy* that he changed his mind on the method of computing these velocities, yet it is also clear that no simple lunar model could account for the entries in this table. In 1974 I proposed a 'skew-equant' model to account for this table of lunar velocity.[16] This 'skew-equant' model adequately accounted

[15] See Goldstein, *Astronomical Tables*, pp. 108–13.

[16] In this 'skew-equant' model the distances from the center of the deferent to the equant and to the observer are not equal: the distance from the equant to the center of the deferent was set equal to 1;22 and the distance from the center of the deferent to the observer was 5;20 (see Goldstein, *Astronomical Tables*, pp. 114–15).

Table 2. *Lunar velocity according to Levi ben Gerson*

I argument (°)	II Levi Δ (°)	III 'Regiom.'/Levi *v*: comp. (°/h)	IV Levi *v*: text (°/h)	V 'Regiom.'/al-B. *v*: comp. (°/h)
0	−0; 4,50	0;29,34	0;29,35	0;29,34
30	−0; 4,20	0;29,55	0;29,57	0;29,56
60	−0; 2,53	0;30,55	0;31, 0	0;31, 0
90	−0; 0,24	0;32,39	0;32,40	0;32,39
120	0; 2,31	0;34,41	0;34,36	0;34,36
150	0; 4,52	0;36,20	0;36,15	0;36,16
180	0; 5,45	0;36,56	0;36,56	0;36,56

The entries in col. II have been derived from Levi's tables (ed. Goldstein, table 35, col. V). The entries in col. III have been computed from those in col. II according to Regiomontanus' rule. The entries in col. IV are taken from Levi's second table of lunar velocity (ed. Goldstein, table 22, col. IV). For purposes of comparison, we display entries in col. V (copied from Table 1, col. V) that are computed according to Regiomontanus' rule based on al-Battānī's correction table.

for the velocities but it had the unfortunate consequence that the maximum lunar correction at syzygy would reach 6;24° instead of Ptolemy's 5;1°. However, it now seems that Levi may not have had recourse to a new model at all, but that he was taking into account the second lunar inequality (represented in Ptolemy's complete lunar model as well as in Levi's complete lunar model). If Levi used Regiomontanus' rule, we should then determine whether he used al-Battānī's values for Δ, or the values in his own table of lunar corrections.[17]

When we compute the values for Δ from Levi's table for lunar anomaly according to his own lunar model for syzygy and apply them in eq. (7), we find the results displayed in Table 2.[18]

It is clear that Levi's table for lunar velocities shows better agreement with the velocities based on al-Battānī's table for the lunar correction (Table 2, col. V, which is copied from Table 1, col. V, above) than with those in Table 2 based on the entries in Levi's own table for the lunar correction ($|T-C| \leqslant 1''$ vs. $|T-C| \leqslant 5''$). Moreover, this method yields better agreement than the

[17] Note that there is a scribal error in the entry for 61° of anomaly in al-Battānī's table: read 4;11,53° instead of 4:11,33°. In the Hebrew version of al-Battānī's tables by Abraham Bar Ḥiyya (twelfth century), we find 4;11,53° (cf. MS Paris, BN heb. 1038, 36b) and this is the value we used in this computation.

[18] For Levi's table of lunar anomaly, see Goldstein, *Astronomical Tables*, Table 35, col. V.

method based on the 'skew-equant' model where $|T - C| \leqslant 3''$. As was true for al-Battānī, Levi does not explain how he computed the entries in his table of velocities. But using the criterion of agreement with recomputation, it would seem that he computed his table with al-Battānī's corrections for lunar anomaly and the formula described by Regiomontanus.

We now turn to the tables of lunar velocity in the corpus of the Alfonsine tables, most of which have not yet been examined in detail:[19]

(i) in a table at degree intervals in MS Oxford, Bodlian Library, abbr. Bodl., Canon Misc. 499, 41v–42r (ascribed to John of Lignères in a later hand),[20] the lunar velocity ranges from $0;29,37^{\circ/h}$ to $0;36,53^{\circ/h}$ (see Table 3, col. IV);

(ii) in a table at 6° intervals ascribed to John of Genoa (MS Paris, Bibliothèque Nationale, abbr. BN, lat. 7282, 129r–129v)[21] but that might belong to John of Lignères (cf. MS Wolfenbüttel 2401, f. 311) the lunar velocity ranges from of $0;29,37,13^{\circ/h}$ to $0;36,58,54^{\circ/h}$, and from $0;11,50,53^{\circ/mn}$ to $0;14,47,33^{\circ/mn}$, where the unit ($^{\circ/mn}$) is degrees per sixtieth of a day (see Table 3, cols. V and VI);

(iii) in a table at degree intervals ascribed to John of Montfort (MSS Paris, BN lat. 7283, 43r–44r, and Oxford, Bodl. Canon. Misc. 499, 151v–152v),[22] the lunar velocities range from $0;11,51,9,11^{\circ/mn}$ to $0;14,47,8^{\circ/mn}$ (see Table 3, col. VII), corresponding to a minimum of $0;29,37,52,57^{\circ/h}$ and a maximum of $0;36,51,57,50^{\circ/h}$.

[19] I wish to thank Dr Donald W. Olsen for drawing my attention to the passage in Poulle, *Les Tables Alphonsines*, pp. 210–11, where these lunar velocity tables are listed, and for supplying me with some preliminary calculations based on a modern analysis of Ptolemy's lunar models showing the effect of the second inequality on the lunar velocity.

[20] I am grateful to Dr John Roche and to Dr Mancha for transcribing this table for me. Dr Mancha also collated this table with another copy that appears on folios 154v–155r in the same manuscript (there were few variants). Note that a second foliation of this manuscript increases by ten the folio numbers used here. This table also appeared in a printed edition: *Tabulae Astronomicae, quas vulgo, ..., Resolutas vocant*...per Ioannem Schonerum (Norimbergae apud Io. Petreium 1536). The entry for 150° is the correct value, 0;36,14, in the edition I consulted: *Opera Mathematica* Ioannis Schoneri (Norinbergae in officina I. Montani & U. Neuberi, 1551), part III, 41r.

[21] I am grateful to Mme Juliane Lay for transcribing the two relevant columns from this MS and from MS Paris, BN lat. 7295A, 137r, I also wish to thank Dr Mancha for providing me with a collation of the hourly lunar velocity table in this MS with the corresponding tables in MSS Paris, BN lat. 7286C, 56v; 7432, 255r; and 7286A, 117r (trivial variants in these MSS have not been recorded in the notes to Table 3). Cf. Goldstein, 'Alfonsine tables', p. 139, where the same maximum and minimum values were reported, based on MS Paris, BN lat. 7295A, 137r.

[22] I am again grateful to Mme Lay for transcribing the Paris copy of this table for me.

Table 3. *A comparison of lunar velocities in the Alfonsine corpus*

I argument (°)	II Δ (°)	III v: comp. (°/h)	IV J. Lignères (°/h)	V John of Genoa (°/h)	VI John of Genoa (°/mn)	VII John of Montfort (°/mn)	VIII v: comp. (°/mn)
0	−0; 4,46	0;29,37,13	–	0;29,37,13	0;11,50,53	–	0;11,50,53
1	4,45	29,37,55	0;29,37	–	–	0;11,51, 9,11*	11,52,10
6	4,43	29,39,18	29,39	29,39,19	11,51,43	11,51,42,38	11,51,43
12	4,39	29,42, 5	29,41	29,42, 6	11,52,50	11,52,49,31	11,52,50
18	4,33	29,46,16	29,45	29,46,17	11,54,31	11,54,29,51	11,54,30
24	4,25	29,51,51	29,51	29,51,51	11,56,44	11,56,43,38	11,56,44
30	−0; 4,14	0;29,59,30	0;29,59	0;29,59,31	0;11,59,48	0;11,59,47,35	0;11,59,48
36	4, 1	30, 8,34	30, 9	30, 8,34	12, 3,25	12, 3,24,58	12, 3,25
42	3,45	30,19,43	30,20	30,19,43	12, 7,53	12, 7,52,31	12, 7,53
48	3,27	30,32,15	30,32	30,32,15	12,12,54	12,12,53,31	12,12,54
54	3, 6	30,46,53	30,47	30,46,53	12,18,45	12,18,44,41	12,18,45
60	−0; 2,43	0;31, 2,54	0;31, 3	0;31, 2,55	0;12,25,10	0;12,25, 9,18	0;12,25,10
66	2,20	31,18,56	31,19	31,18,55	12,31,34	12,31,33,55	12,31,34
72	1,53	31,37,44	31,38	31,37,45	12,39, 6	12,39, 5,25	12,39, 6
78	1,25	31,57,15	31,57	31,57,15	12,46,54	12,46,53,38	12,46,54
84	0,57	32,16,45	32,17	32,16,45	12,54,42	12,54,41,51	12,54,42
90	−0; 0,24	0;32,39,44	0;32,40	0;32,39,45	0;13, 3,54	0;13, 3,53,41	0;13, 3,54
96	+0; 0,13	33, 5,31	33, 5	33, 8,31*	13,14,13	13,14,12,25	13,14,12
102	0,39	33,23,38	33,24	33,30,36*	13,21,28	13,21,27,11	13,21,27
108	1,16	33,49,24	33,49	33,49,21*	13,31,46	13,31,45,55	13,31,46
114	1,46	34,10,18	34,10	34,10,19	13,40, 8	13,40, 7,35	13,40, 7
120	0; 2,22	0;34,35,23	0;34,36	0;34,35,24	0;13,50,10	0;13,50, 9,35	0;13,50, 9

126	2,55	34,58,23	34,58	34,58,23	13,59,58*	13,59,21,25	13,59,21
132	3,24	35,18,35	35,19	35,18,25*	14, 7,47*	14, 7,26,21	14, 7,25
138	3,53	35,38,47	35,39	35,38,47	14,15,32	14,15,31,18	14,15,31
144	4,20	35,57,36	35,58	35,57,36	14,23, 3	14,23, 2,48	14,23, 2
150	0; 4,43	0;36,13,37	0;36,18*	0;36,13,37	0;14,29,27	0;14,30, 0,51*	0;14,29,27
156	5, 4	36,28,15	36,28	36,28,15	14,35,18	14,35,18,35	14,35,18
162	5,18	36,38, 0	36,38	36,38, 0	14,39,12	14,39,12,41	14,39,12
168	5,30	36,46,22	36,46	36,46,22	14,42,33	14,42,33,21	14,42,33
174	5,37	35,51,14	36,51	36,53,15*	14,45,18*	14,44,30,25	14,44,30
180	+0; 5,40	0;36,53,20	0;36,53	0;36,58,54*	0;14,47,33*	0;14,44,47, 8*	0;14,45,20

The entries in col. IV have been copied from MS Oxford, Canon Misc. 499, 41v–42r; in cols. V and VI from MS Paris, BN lat. 7282, 129r–129v; and in col. VII from MS Paris, BN lat 7283, 43r–44r.

Col. IV, 150. Read: 0;36,**14** (with MS Oxford, Canon Misc. 499, 155r).
Col. V, 96. Read: 33,**5**,31 (with MS Paris, BN lat. 7286A).
Col. V, 102. Read: 33,**23**,**37** (with MS Paris, BN lat. 7286A).
Col. V, 108. Read: 33,49,**35** (with MSS Paris, BN lat. 7432 and 7295A).
Col. V, 132. Read: 35,18,**35** (with MSS Paris, BN lat. 7432 and 7295A).
Col. V, 174. See the discussion at the end of Section III.
Col. V, 180. See the discussion at the end of Section III.
Col. VI, 126. Read: 13,59,**21**.
Col. VI, 132. Read: 14,7,**27**.
Col. VI, 174. See the discussion at the end of Section III.
Col. VI, 180. See the discussion at the end of Section III.
Col. VII, 1. See the discussion at the end of Section III.
Col. VII, 150. The discrepancy here seems to be due to the computation and not to a copyist's error.
Col. VII, 180. See the discussion at the end of Section III.

Since c (and hence Δ) is only given to seconds in the Alfonsine tables, the introduction of sexagesimal thirds and fourths in tables of lunar velocity is unwarranted. In Table 3, col. II, we display the values for Δ computed from the entries for the lunar corrections in the Alfonsine tables.[23] To compute the velocities according to the Alfonsine tables, we first compute a value for v_α in eq. (10), above, based on the Alfonsine parameters: $\bar{v}_\alpha = 0;32,39,44,54^{\circ/\mathrm{h}}$, $\bar{v}_{2\eta} = 1;0,57,13,28^{\circ/\mathrm{h}}$, and $\Delta c' = 0;9^\circ$.[24] Thus,

$$\begin{aligned} v_\alpha &= \bar{v}_\alpha + \bar{v}_{2\eta} \cdot \Delta c' \\ &= 0;32,39,44,54 + 1;0,57,13,28 \cdot 0;9 = 0;41,48,19,55 \\ &\approx 0;41,48. \end{aligned}$$

Then we compute v from eq. (8), above, where $\bar{v} = 0;32,56,27,33^{\circ/\mathrm{h}}$:[25] the results in degrees per hour are displayed in Table 3, col. III, and the corresponding entries in degrees per sixtieth of a day are displayed in col. VIII.

It is clear that the agreement of the computed values with the entries in all versions of these Alfonsine lunar velocity tables is generally quite good, and that there are only a few problematic cases (these are marked with an asterisk in Table 3). Unfortunately, these problematic cases affect the minimum and maximum values that one might well consider as characterizing these tables. Moreover, while it is clear that the three versions of these tables are very similar, their exact relationship has not yet been determined.

In Table 3, col. IV, a problem occurs at 150°, but this is an isolated error in one copy only; otherwise $|\mathrm{T}-\mathrm{C}| \leqslant 1''$. In cols. V and VI, the entries in the text corresponding to 174° and 180° agree with each other, but not with recomputation: I have no explanation to offer for this.[26] In col. VII, there is excellent agreement with the recomputed values (i.e. $|\mathrm{T}-\mathrm{C}| \leqslant 1'''$, where the entries in the table are rounded to thirds) except for 150° and at the beginning and end of the table: there is certainly confusion near 0° and 180° (in fact, near minimum the entries for 1° and 3° are identical and the entries for 2°, 4° and 5° are all 0;11,51,25,55, while near maximum the entries for 174° to 176° are all identical as are the entries for 177° to 180°). So, here the maximum and minimum values are less informative than one might expect.

[23] See Poulle, *Les Tables Alphonsines*, pp. 148–53, col. 5.

[24] See *ibid.*, pp. 136, 139.

[25] See *ibid.*, p. 135.

[26] A similar problem afflicts one of the other lunar velocity tables in the Alfonsine corpus: cf. Goldstein, 'Alfonsine tables', I am informed by Dr Mancha that in MS Paris, BN lat. 7286A, 117r, in a table for lunar velocity not ascribed to John of Genoa but significantly related to his table, the entry for 174° is 0;36,51,15, which is much closer to the recomputed value. It is possible that a scribal error occurred in the copy of the Alfonsine table of lunar anomaly used to compute these entries.

IV

We have argued that the second lunar inequality was used in computing lunar velocity by Levi ben Gerson. Moreover, despite the discrepancies among the three versions of lunar velocity tables in the Alfonsine corpus discussed above, the agreement found with recomputation is sufficient to allow us to conclude that the entries in all three versions also take into account the second lunar inequality. Since Levi ben Gerson and the Alfonsine astronomers were active in the 1330s, it is not possible to assign priority and, on the basis of sources currently available, one cannot say if they knew of each other's work.

From the preceding remarks, it would seem that Regiomontanus' rule was already known in the fourteenth century. Yet, in the absence of direct testimony, a note of caution should be added. Was there another way to compute these lunar velocities available at the time? We have already mentioned the usefulness of the method described in Appendix 2 for computing lunar velocities according to Ptolemy's simple lunar model. Obviously, this method can be modified to take into account any other lunar model. In fact, there is a fourteenth-century text which seems to tell us to do something very much like that: John of Saxony's canons for the Alfonsine tables, composed in 1327.[27] In chapter 22, concerning the time from mean syzygy to true syzygy, there is an instruction for computing lunar velocity as the progress of the Moon in 1/60 of a day (i.e., $0;24^{h}$).[28] John of Saxony does not specify which lunar model should be used for computing these positions 1/60 of a day apart but, in presenting a worked example, Poulle reasonably assumed that the complete lunar model was intended.[29] The same argument would hold for the lunar velocity tables of Levi ben Gerson as well.

Thus, we have two alternative methods for explaining the computation of lunar velocity tables in the fourteenth century, and more evidence may be needed to decide between them.[30]

27 Poulle, *Les Tables Alphonsines*, p. 17.

28 *Ibid.*, pp. 84–5. Dr Mancha informs me that an almost identical method is reported in John of Genoa's *Canones eclipsium* dated 1332 and preserved in at least 3 MSS (Paris, BN lat. 7322 and 7281; Oxford, Digby 97). John of Genoa gives rules for calculating positions of the Moon at an interval of 1 hour rather than 1/60 day as in John of Saxony's canons.

29 Poulle, *Les Tables Alphonsines*, pp. 217–18.

30 Recently, Dr Mancha has found evidence to suggest that the computation of lunar velocity as the progress of the Moon in a given time interval may go back at least to John of Sicily's commentary on the canons for the Toledan tables, composed *ca.* 1290 (see Paris, BN lat. 7281, ff. 46r–138r, esp. f. 111r), and this remains to be explored.

Appendix 1

Pederson presents the following argument (translated into our notation) to support his claim that the second lunar inequality cannot be ignored in computing the lunar velocity at syzygy.[31] Let

$$\lambda = \bar{\lambda} + e(2\eta, \alpha')$$

where e is the total correction (or 'equation') due to both the first and the second inequalities. Then

$$v = \bar{v} + \frac{de(2\eta, \alpha')}{dt}$$

or

$$v = \bar{v} + \frac{\partial e}{\partial(2\eta)} \cdot \bar{v}_{2\eta} + \frac{\partial e}{\partial \alpha'} \cdot v_{\alpha}. \qquad \text{(a1)}$$

At syzygy the second term vanishes. As for the third term,

$$v_{\alpha} = \bar{v}_{\alpha} + \frac{dc'}{dt}$$

or

$$v_{\alpha} = \bar{v}_{\alpha} + \frac{dc'}{d(2\eta)} \cdot \bar{v}_{2\eta}. \qquad \text{(a2)}$$

Therefore, $v_{\alpha} = \bar{v}_{\alpha}$ only when the second term in eq. (a2) is 0: this is true for Ptolemy's simple model, but not for his complete model (even at syzygy).

Appendix 2

We present the following method for computing the lunar velocity.[32] To find the hourly lunar velocity at anomaly α, let

$$\alpha_1 = \alpha - (\bar{v}_{\alpha}/2)$$

and

$$\alpha_2 = \alpha + (\bar{v}_{\alpha}/2)$$

i.e., α_1 is the anomaly at a half hour before α, and α_2 is the anomaly at a half hour after α. Then

$$\lambda_1 = \bar{\lambda}_1 + c(\alpha_1)$$

and

$$\lambda_2 = \bar{\lambda}_2 + c(\alpha_2).$$

[31] Pedersen, *Almagest*, p. 226. [32] Goldstein, *Astronomical Tables*, pp. 111–12.

But

$$\bar{\lambda}_2 = \bar{\lambda}_1 + \bar{v}.$$

Hence,

$$\lambda_2 - \lambda_1 = \bar{v} + c(\alpha_2) - c(\alpha_1).$$

and this is the progress of the Moon in an hour which we assign to anomaly α. This method yields the same results, to sexagesimal thirds, as the precise determination by means of the modern formula for the instantaneous lunar velocity at anomaly α in Ptolemy's simple model; but it avoids introducing the concept of 'instantaneous' velocity.