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Volterra Integral and Functional Equations

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Volterra Integral and Functional Equations

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PREFACE

During the past 25 years the theories of Volterra integral equations, Volterra integrodifferential equations, and functional differential equations have undergone rapid developments. What began as a few scattered papers on specific equations, and on particular applied problems, has grown to branches of applied analysis of considerable size, having rich structures of their own. The growth has been strongly promoted by the large number of applications that these theories have found in physics, engineering, and biology. Our understanding of those comparatively simple, usually linear, problems that were present from the beginning has increased significantly. In addition, knowledge has been obtained about more general and about more complex equations. We observe, for example, that the main part of the asymptotic theory for nonlinear Volterra integral and integrodifferential equations is of fairly recent origin. The same observation can be made about equations involving parameters or additional variables, and more generally about equations in infinite-dimensional spaces.

It is very common to make a fairly sharp distinction between Volterra integral and integrodifferential equations on the one hand, and functional differential equations on the other hand. In many respects this distinction is artificial, being due more to the different backgrounds of the researchers than to any inherent differences. In the sequel we refer to these groups of equations as 'Volterra integral and functional equations', or 'Volterra equations' for short, and many of the results that we give apply equally well to functional differential equations and to Volterra integral or integrodifferential equations. However, our general attitude to these equations resembles the attitude that has been prevailing for the last few years among people working in Volterra integral and integrodifferential equations. In particular, we pay no attention to questions that are specific for finite delay problems.

Although our knowledge of Volterra equations is swiftly growing, the point may well be made that the study of Volterra equations in finite-dimensional spaces has reached a certain maturity, which both motivates and makes possible a coherent presentation. This book testifies to the fact that we share this opinion. Certainly, we do not believe that Volterra equations in $\mathbf{R}^n$ or $\mathbf{C}^n$ constitute a closed chapter. It suffices to consider the scarcity of results on periodic solutions of nonlinear equations, or the so far unsatisfactory resolvent theory for nonintegrable kernels, to realize that the field is wide open for progress. But we do believe that the available research results make up a certain whole which may profitably be presented in the form of a book. In addition, this will make articles scattered throughout the literature readily available to the working mathematician.

The class of ordinary differential equations is subsumed under the class of Volterra equations. However, the importance of this fact should not be exaggerated. The present text makes it clear that the theory of Volterra equations exhibits a rich variety of features not present in the theory of ordinary differential equations. The converse statement also holds: some of the more specific results available for ordinary differential equations cannot possibly be extended to Volterra equations.

Chapter 1 gives an introduction to Volterra equations and a general overview of the contents. The remaining part of the book comprises 19 chapters and is divided into three parts. In Part I (Chapters 2–10) we consider the linear theory, and in Part II (Chapters 11–15) we deal with quasilinear equations and existence problems for general nonlinear equations, and give some general asymptotic results. Part III (Chapters 16–20) is devoted to frequency domain methods in the study of nonlinear equations.

Obviously, to some extent the contents and the general approach to the subject reflect the research interests of the authors. This is particularly true of Chapters 15–20. In fact, a presentation of the results in Part III provided the initial impetus for the writing of this book. It was soon realized, however, that a treatment of the linear theory, together with the quasilinear and existence chapters of Part II, was equally important. The project thus grew into an exposé of both linear and nonlinear Volterra equations in finite-dimensional spaces. Within these limits, we have attempted to incorporate as much as possible of what we feel is essential to the understanding of Volterra equations. Where limitation of space has prohibited the inclusion of the complete proof, we have tried to give at least an outline. Additional results are frequently touched upon in the comment sections at the end of each chapter.

The major part of the text is made up of results that have so far at most been available in research papers. In particular, this is true for Chapters 4–10, 12, and 15–20. A number of the results given here have not appeared in print before, and many previously existing results have been modified and improved.

At an early stage it was decided not to include a number of worthy topics in Volterra equations: equations in infinite-dimensional spaces, stochastic equations, geometric and degree theory, attractors, stable and unstable manifolds, eigenfunction expansions, bifurcation theory, control and optimization problems, oscillation results, and numerical methods. Very little is said about neutral equations. The primary motivation for these omissions is simply a question of volume. There is more than enough in the basic theory of Volterra equations in finite-dimensional spaces to fill a good-sized book. A particular motivation was that many of these fields, for example abstract Volterra equations, are still in a state of flux, with new basic results continuously emerging.

With a few exceptions, the entire text analyses n-dimensional rather than scalar equations. This, of course, gives the results a greater generality and a wider applicability, and it facilitates generalizations to infinite-dimensional spaces. In Parts I and II this setting is quite natural, and it does not overly complicate the proofs. The n-dimensional results in Chapters 16–19 have appeared earlier in the literature almost exclusively in a scalar setting. It does, however, appear natural and consistent to present the nonlinear asymptotic theory in the same general n-dimensional setting as we use in the earlier chapters. The theory of scalar kernels of positive type can be extended to matrix-valued kernels without too much additional complexity, and this is done in Chapter 16. In some of the theorems in Chapters 17–19 one has to make the significant additional assumption that the nonlinearity is the gradient of a scalar-valued function; when $n = 1$ this assumption is automatically satisfied.

A substantial part of the text concerns equations with measure-valued kernels. In particular, this is true for Chapters 3, 4, 10, 16, and for parts of Chapters 17–19. Our results may therefore be applied to delay and differential-delay equations. The approach that we take to functional differential equations differs, at least in appearance, from the common one. The main difference is that we stress resolvent theory, and avoid semigroup techniques. In Chapter 8 a semigroup theory is presented briefly, but the topic is not pursued further. One consequence of our approach is that there is no need for a separate treatment of functional differential equations with finite delay. Hence, we do not discuss typical finite delay questions, such as eigenfunction expansions, solutions that vanish after a finite time, and backward continuation of solutions. Instead, we attach primary importance to the integrability properties of the kernels and the resolvents. Also, we put the emphasis on a study of the Laplace transform of the kernel instead of on a study of the spectrum of the generator of the semigroup (which, however, in reality amounts to the same thing).

Every effort has been made to achieve a reasonably self-contained presentation. We have been forced to assume some basic knowledge of analysis, and do occasionally fall back on results in Rudin [2] and [3], Hille and Phillips [1], and Hewitt and Stromberg [1]. Even so, a significant part of the text has been devoted to analysis results which are difficult to find in the literature, but which are needed for the development of the theory. In particular, we present a number of theorems on vector measures, on convolutions, and on Laplace transforms. These results have been separated from the main body of the text, and several chapters contain appendices describing results of a technical nature.

Exercises and some comments on further references, historical developments, etc., are included at the end of each chapter. We warn the reader that the exercises cover a wide range of difficulty. Some of the problems are easy applications of the text, others require a much deeper understanding of the topic.

One of our intentions when writing this book was to stimulate further research. Consequently, we have tried to make the text reflect as much as possible of the present state of knowledge in the area of the theory of Volterra equations that we cover. In the case where a result included in the text is not the best available, we have tried to incorporate a statement to this effect, together with further references in the corresponding comment section.

We sustain a hope that the text will bring about more interaction between research workers in the two fields of functional differential equations on the one hand and Volterra integral and integrodifferential equations on the other hand. We believe that both areas, which have been unduly separated from each other, would profit from such an interaction. In addition, we hope that an even larger number of concrete applications would find their way into the present theory of Volterra equations. Without the feedback from and the incitement supplied by such applications in physics, engineering, and biology, the future of the field would look much bleaker.

This text was composed and typeset at the Institute of Mathematics of the Helsinki University of Technology. It is a pleasure for us to acknowledge the excellent working conditions and facilities at our Institute that made this voluminous venture possible. At various stages of our efforts the Academy of Finland relieved some of us from teaching and administrative duties. Needless to say, we appreciate this support. Visits to Madison, Blacksburg, Graz, and Carnegie-Mellon also contributed to the final product.

Over the years John A. Nohel has actively encouraged us in our endeavour. We express our gratitude for his continuous and enthusiastic support. Our colleagues at Virginia Tech, Kenneth B. Hannsgen and Robert L. Wheeler, have given us much time and advice. In particular we wish to thank them for their fruitful criticism of earlier versions of the manuscript. Our thanks also go to Ismo Sedig for comments on Chapter 15.

We are very pleased to see our work appear in the series 'Encyclopedia of Mathematics' of the Cambridge University Press and the pleasant cooperation with David Tranah is gratefully acknowledged.

Our wives have undauntedly endured several years of book-writing with subsequent neglect of family duties and delights. All things considered, without their persistent support and teasing comments this book would never have been finished.

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Olof Staffans

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LIST OF SYMBOLS

Basic Sets and Spaces

$\mathbf{R}$	$(-\infty, \infty)$.
$\mathbf{R}^+$	$[0, \infty)$.
$\mathbf{R}^-$	$(-\infty, 0]$.
$\mathbf{C}$	The complex plane.
$\mathbf{E}$	Either $\mathbf{R}$ or $\mathbf{C}$.
$\mathbf{Z}$	The set of integers.
$\mathbf{N}$	The set of nonnegative integers.
$\mathbf{R}^n$	The set of n-dimensional column vectors with real entries; various norms and inner products.
$\mathbf{C}^n$	The set of n-dimensional column vectors with complex entries; various norms and inner products.
$\mathbf{R}^{n\times m}$	The set of $(n \times m)$-dimensional matrices with real entries; a norm adapted to the corresponding norms in $\mathbf{R}^m$ and $\mathbf{R}^n$.
$\mathbf{C}^{n\times m}$	The set of $(n \times m)$-dimensional matrices with complex entries; a norm adapted to the corresponding norms in $\mathbf{C}^m$ and $\mathbf{C}^n$.
$\mathcal{T}_S$	The real line $\mathbf{R}$ where the points $t+mS$, $m \in \mathbf{Z}$, are identified.
$\emptyset$	The empty set.

Function Spaces

$V(J;Q)$	Functions of type V ($= L^p$, BC etc.) with domain J and range contained in Q ($= \mathbf{R}^n$, $\mathbf{C}^{n\times n}$ etc.).
$V(J;\eta;Q)$	Functions of type V ($= L^p$, BC etc.) with domain J, weight function η, and range contained in Q.
$V_{\mathrm{loc}}(J;Q)$	Functions of type V with domain J that belong to $V(K;Q)$ for every compact subset K of J. If $V(K;Q)$ is a Banach space, then $V_{\mathrm{loc}}(J;Q)$ is given the topology induced by the

	metric $d(\psi,\phi)=\sum_{j=1}^{\infty}\{2^{-j}\|\psi-\phi\|_{V(K_j)}/(\|\psi-\phi\|_{V(K_j)}+1)\}$ where $\bigcup_{j=1}^{\infty}K_j=J$.
$V(\mathcal{T}_S;Q)$	S-periodic functions whose restrictions to each finite interval are of type V with range in Q; $\|\phi\|_{V(\mathcal{T}_S)}=\|\phi\|_{V([0,S))}$.
$U \mathbin{\dot{+}} V$	The space of functions of the type $\{\, f+g \mid f\in U,\ g\in V\,\}$, where U and V are some function spaces with $U\cap V=\{0\}$.
AC	Absolutely continuous functions.
AP	Almost-periodic functions; sup-norm.
B^∞	Borel measurable, bounded functions; sup-norm.
B_0^∞	Functions in B^∞ that tend to zero at infinity.
B_ℓ^∞	Functions in B^∞ with limits at infinity (not necessarily the same limit at $-\infty$ and $+\infty$ if the domain is $\mathbf{R}$).
B_{loc}^∞	Functions locally in B^∞, i.e., their restrictions to compact subsets of the domain belong to B^∞; see $V_{\mathrm{loc}}(J,Q)$ above.
$BBV(J)$	Functions in $BV_{\mathrm{loc}}(J)$, where $J=\mathbf{R}$ or $\mathbf{R}^+$, satisfying $\|\phi\|_{BBV(J)}=\sup_{[s,s+1]\subset J}\|\phi\|_{BV([s,s+1])}<\infty$.
$BBV_0(J)$	Functions in $BBV(J)$ such that $\|\phi\|_{BV([s,s+1])}$ tends to zero at infinity. Here $J=\mathbf{R}$ or $\mathbf{R}^+$.
BC	Bounded continuous functions; sup-norm.
BC_0	Functions in BC that tend to zero at infinity.
BC_ℓ	Functions in BC with limits at infinity (not necessarily the same limit at $-\infty$ and $+\infty$ if the domain is $\mathbf{R}$).
$BL^p(J)$, $1\le p\le\infty$:	Functions in $L^p_{\mathrm{loc}}(J)$ with a finite norm $\|\phi\|_{BL^p(J)}=\sup_{(s,s+1)\subset J}\|\phi\|_{L^p(s,s+1)}<\infty$. Here $J=\mathbf{R}$ or $\mathbf{R}^+$.
$BL^p_0(J)$, $1\le p\le\infty$:	Functions in $BL^p(J)$ such that $\|\phi\|_{L^p(s,s+1)}$ tends to zero at infinity. Here $J=\mathbf{R}$ or $\mathbf{R}^+$.
BUC	Bounded uniformly continuous functions; sup-norm.
BUC^1	Bounded uniformly continuous functions with bounded uniformly continuous derivative.
BV	Functions of bounded variation; the norm is the sum of the total variation and the sup-norm.
BV_{loc}	Functions locally of bounded variation, i.e., their restrictions to compact subsets of the domain are of bounded variation, see $V_{\mathrm{loc}}(J,Q)$ above.
C	Continuous functions; sup-norm on each compact set. The same space as BC_{loc}.
C_c	Continuous functions with compact support.
C_c^∞	Infinitely many times differentiable functions with compact support.

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L^p, $1 \le p < \infty$:	Measurable functions with finite norm $\left\{\int \lvert\phi(t)\rvert^p \, \mathrm{d}t\right\}^{1/p}$.
L^∞	Measurable functions with finite norm $\operatorname{ess\,sup}\lvert\phi(t)\rvert$.
L^∞_0	Functions in L^∞ with an essential limit 0 at infinity.
L^∞_ℓ	Functions in L^∞ with essential limits at infinity (not necessarily the same limit at $-\infty$ and $+\infty$ if the domain is $\mathbf{R}$).
L^p_{loc}, $1 \le p \le \infty$:	Functions locally in L^p, i.e., their restrictions to compact subsets of the domain belong to L^p; see $V_{\mathrm{loc}}(J, Q)$ above.
$W^{1,p}$, $1 \le p \le \infty$:	Locally absolutely continuous functions that together with their derivative belong to L^p; norm $\lVert\phi\rVert_{W^{1,p}(J)} = \lVert\phi\rVert_{L^p(J)} + \lVert\phi'\rVert_{L^p(J)}$.
$\tilde{L}^1$, $\hat{L}^1$, $\widehat{M}$	Functions that are Fourier or Laplace transforms of L^1-functions or finite measures; see Sections **6**.3 and **7**.3.
$\mathcal{D}$	Infinitely many times differentiable functions with compact support.
$\mathcal{D}'$	Distributions; the dual of $\mathcal{D}$.
$\mathcal{S}$	Infinitely many times differentiable functions that decay rapidly at infinity.
$\mathcal{S}'$	Tempered distributions; the dual of $\mathcal{S}$.
$\mathcal{V}$	Volterra kernels of various types; see Chapters **9** and **10**.
$\mathcal{F}$	Fredholm kernels of various types; see Chapter **9**.
$\mathcal{K}$	This letter stands for one of the letters $\mathcal{F}$ or $\mathcal{V}$.
$[V; J]$	Kernels of type V on J.

Measure Spaces

$M(J; Q)$	Finite measures on J with range contained in Q and total variation norm; see Section **3**.2.
$M(J; \eta; Q)$	Measures on J with range contained in Q and total variation norm weighted by the function η; see Section **4**.3.
$M_{\mathrm{loc}}(J; Q)$	Set functions on J that belong to $M(K; Q)$ for every compact subset K of J.
PT	Kernels of positive type; see Chapter **16**.
PT_{aco}	Kernels of anti-coercive type; see Section **16**.5.
$\mathcal{M}$	Nonconvolution measure kernels of various types; see Chapter **10**.

Transforms

$\hat{\mu}$	Laplace–Stieltjes transform of the measure μ, $\hat{\mu}(z) = \int_{\mathbf{R}^+} e^{-zs}\mu(ds)$ if μ is measure on $\mathbf{R}^+$, and $\hat{\mu}(z) = \int_{\mathbf{R}} e^{-zs}\mu(ds)$ if μ is a measure on $\mathbf{R}$.
$\tilde{\mu}$	Fourier–Stieltjes transform of the measure μ, $\tilde{\mu}(\omega) = \hat{\mu}(i\omega)$.
$\hat{a}$	Laplace transform of the function a, $\hat{a}(z) = \int_0^\infty e^{-zs}a(s)\,ds$ if a is defined on $\mathbf{R}^+$, and $\hat{a}(z) = \int_{-\infty}^\infty e^{-zs}a(s)\,ds$ if a is defined on $\mathbf{R}$.
$\tilde{a}$	Fourier transform of the function a, $\tilde{a}(\omega) = \hat{a}(i\omega)$.

Operators and various other Symbols

$\langle u, v\rangle$	Inner product of u and v.
$\lvert v\rvert$	Absolute value of v if v is a scalar; norm of v if v is a vector or a matrix.
$\lvert\mu\rvert(E)$	Total variation measure, $\sup\sum_{j=1}^N \lvert\mu(E_j)\rvert$, where $\{E_j\}_{j=1}^N$ is a partition of E.
$\lvert v\rvert_+$	$\max\{0, v\}$. Here v is real.
$\lvert v\rvert_-$	$\max\{0, -v\}$. Here v is real.
$\lVert\phi\rVert_V$	Norm of ϕ in the space V. Here ϕ is a function or a measure.
$\lVert\phi\rVert_{\sup(J)}$	Supremum of $\lvert\phi\rvert$ on J.
$\lVert\phi\rVert_{\mathrm{var}(J)}$	Total variation of ϕ on J.
$\vert\vert\vert k\vert\vert\vert_{L^p(J)}$	$\sup_{\substack{\lVert g\rVert_{L^q(J)}\le 1\\ \lVert f\rVert_{L^p(J)}\le 1}} \int_J\int_J \lvert g(t)k(t,s)f(s)\rvert\,ds\,dt$, $1/p + 1/q = 1$; see Section **9**.2.
$\vert\vert\vert k\vert\vert\vert_{L^{p,q}(J)}$	$\sup_{\substack{\lVert f\rVert_{L^p(J)}\le 1\\ \lVert g\rVert_{L^{q'}(J)}\le 1}} \int_J\int_J \lvert g(t)k(t,s)f(s)\rvert\,ds\,dt$, $1/q + 1/q' = 1$; see Section **9**.2.
$\vert\vert\vert k\vert\vert\vert_{B^\infty(J)}$	$\sup_{t\in J}\int_J \lvert k(t,s)\rvert\,ds$; see Section **9**.5.
$\vert\vert\vert \kappa\vert\vert\vert_{B^\infty(J)}$	$\sup_{t\in J}\lvert\kappa\rvert(t,J)$; see Section **10**.2.
$\vert\vert\vert \kappa\vert\vert\vert_{[B^\infty;L^1](J)}$:	$\int_J \lvert\kappa\rvert(t,J)\,dt$; see Section **10**.3.
$\vert\vert\vert k\vert\vert\vert_{[M;B^\infty](J)}$:	$\sup_{t,s\in J}\lvert k(t,s)\rvert$; see Section **10**.3.
$\Re$	The real part, or the symmetric part of a matrix.
$\Im$	The imaginary part, or the anti-symmetric part of a matrix.
$\sigma(\varphi)$	The spectrum of the function φ; see Section **15**.4.
$\Gamma(\varphi)$	The limit set of the function φ; see Section **15**.2.
τ_h	Translation operator: $(\tau_h\varphi)(t) = \varphi(t+h)$.

$*$	Convolution (see Sections **2**.2, **3**.2 and **4**.1).
$\star$	The convolution-like product defined in Sections **9**.2, **10**.2 and **10**.3.
$\succeq$	$A \succeq 0$ if $\langle y, Ay \rangle \geq 0$ for all $y \in \mathbf{C}^n$, where $\langle \cdot, \cdot \rangle$ is some given inner product; $u \succeq v$ if $u - v \in K$ where K is a given cone.
$\succ$	$A \succ 0$ if $\langle y, Ay \rangle > 0$ for all $y \in \mathbf{C}^n \setminus \{0\}$; $u \succ v$ if $u - v$ belongs to the interior of a given cone K.
$\overline{z}$	Complex conjugate of z.
$\overline{\Omega}$	Closure of the set Ω.
$\alpha\{\Omega\}$	In Chapter **12**, the measure of noncompactness of the set Ω.
$\text{conv}\{\Omega\}$	Convex hull of the set Ω.
$\text{diam}\{\Omega\}$	Diameter of the set Ω; $\text{diam}\{\Omega\} = \sup\{ \|x-y\| \mid x, y \in \Omega \}$.
A^*	The adjoint of the matrix A with respect to some given inner product.
$\det[A]$	Determinant of the matrix A.
$\text{diag}[\alpha_1, \ldots, \alpha_n]$:	Diagonal matrix with entries $\alpha_1, \ldots, \alpha_n$.
$\ker[A]$	Kernel (nullspace) of the matrix A.
$\text{range}[A]$	Range of the matrix A.
$\text{span}\{\Omega\}$	Linear span of the set Ω of vectors.
ln	Logarithm with base e.
lim	Limit.
lim sup	Superior limit.
lim inf	Inferior limit.
inf	Infimum.
sup	Supremum.
ess inf	Essential infimum.
ess lim	Essential limit.
ess sup	Essential supremum.
$[\nu, \phi]$	The value of the distribution ν evaluated at the test function ϕ.
$[\![\mu, \phi]\!]$	$\Re \int_{\mathbf{R}} \langle \phi(t), (\mu * \phi)(t) \rangle \, dt$; see Section **16**.2.
δ_t	Unit point mass at t, $\delta = \delta_0$.
μ_a	Absolutely continuous part of the measure μ.
μ_d	Discrete part of the measure μ.
μ_s	Singular part of the measure μ.
χ_E	Characteristic function of the set E.
I	Identity matrix.

$Q(\phi, \mu, T)$	$\int_0^T \langle \phi(t), \int_{[0,t]} \mu(\mathrm{d}s)\phi(t-s) \rangle \,\mathrm{d}t$, used in Chapters **17**–**19**.
$T_{\max}$	In Chapter **12**, the upper limit for the interval of existence of a noncontinuable solution.
x_M	In Section **13**.4, maximal solution.
x_m	In Section **13**.4, minimal solution.