1

Introduction and Overview

We give a brief introduction to Volterra equations, review some simple mathematical models involving such equations, and give an overview of the contents of the book.

1. Introduction

Let $T(\xi, t)$ be the temperature in a semi-infinite ($\xi \geq 0$) one-dimensional bar. Assume that the bar loses energy at a rate proportional to $T(0,t)$ at the point $\xi = 0$, and that an external source generates heat proportional to the function $q(t)$ at this end of the bar. If the bar is insulated at all other parts, and has temperature zero at time zero, then T is a solution of the following initial-boundary value problem (where some proportionality constants have been set equal to 1):

$$\left.\begin{aligned} T_t(\xi,t) &= T_{\xi\xi}(\xi,t), && \xi > 0, \quad t > 0,\\ T_\xi(0,t) &= \alpha T(0,t) - q(t), && \qquad\quad t > 0,\\ T(\xi,0) &= 0, && \xi \geq 0,\\ \lim_{\xi\to\infty} T(\xi,t) &= 0, && \qquad\quad t \geq 0. \end{aligned}\right\} \tag{1.1}$$

Using Laplace transforms it is not hard to show that T satisfies the equation

$$T(\xi,t) = \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}} \mathrm{e}^{-\xi^2/(4(t-s))} \big(q(s) - \alpha T(0,s)\big)\,\mathrm{d}s, \quad t > 0, \quad \xi \geq 0.$$

Consequently, the function T can be obtained once $T(0,t)$ is known. Taking $\xi = 0$ above, we get the convolution Volterra integral equation

$$x(t) + \int_0^t k(t-s)x(s)\,\mathrm{d}s = f(t), \quad t \geq 0, \tag{1.2}$$

where $x(t) = T(0,t)$, $k(t) = \frac{\alpha}{\sqrt{\pi}} t^{-\frac{1}{2}}$, and $f(t) = \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}} q(s)\,\mathrm{d}s$.

Equation (1.2) is a typical example of the equations to be studied in this book. Of course, one might just as well write the equation in the form

$$x(t) = \int_0^t h(t-s)x(s)\,\mathrm{d}s + f(t), \quad t \geq 0,$$

where $h(t) = -k(t)$, but we shall usually write the integral on the left-hand side. (Observe that, in this example, the forcing function f is of the special form $\int_0^t k(t-s)\phi(s)\,\mathrm{d}s$. In some of our results we shall take advantage of this fact.) In (1.2), the unknown function $x(t)$ appears both by itself and under the integral sign multiplied by the kernel k. This fact qualifies this equation as a Volterra equation of the second kind.

As we will see later, the existence of a solution of an equation of the second kind usually presents less of a problem. A more difficult and challenging task is to determine how the solution behaves as $t \to \infty$.

As another example, let us briefly consider a control system with negative, delayed feedback. The simplest model for such a system is the differential-delay equation

$$\begin{aligned} x'(t) &= -ax(t-\tau), && t \geq 0; \\ x(t) &= \varphi(t), && t \in [-\tau, 0], \end{aligned} \tag{1.3}$$

where $\tau > 0$. One possible approach to solve this equation is to use semigroup theory, but in this book we will mostly take this equation as an example of the integrodifferential equation

$$x'(t) + \int_{[0,t]} \mu(\mathrm{d}s)x(t-s) = f(t), \quad t \geq 0; \quad x(0) = x_0. \tag{1.4}$$

To get (1.3) from (1.4), let μ be a point mass with weight a at the point τ, i.e., $\mu(E) = a$ if $\tau \in E$; otherwise $\mu(E) = 0$. Furthermore, define f by $f(t) = -a\chi_{[0,\tau)}(t)\varphi(t-\tau)$, and let $x_0 = \varphi(0)$.

A common way of writing a linear functional differential-delay equation is

$$\begin{aligned} x'(t) &= \int_{(-\infty,0]} \eta(\mathrm{d}s)x(t+s) + F(t), && t \geq 0; \\ x(t) &= \varphi(t), && t \leq 0, \end{aligned}$$

where the initial function φ is prescribed. However, by defining $f(t) = F(t) + \int_{(-\infty,-t)} \eta(\mathrm{d}s)\varphi(t+s)$ and writing $\mu(E) = -\eta(-E)$, this equation may as well be reduced to (1.4).

We shall frequently use the fact that (1.4) can be put into the form (1.2) by integration.

The two preceding examples share one property: neither of them can be reduced to a finite-dimensional ordinary differential equation. Another common feature is that, in both cases, the feedback is time invariant, i.e., the influence of the past values $x(s)$ on $x(t)$ (or on $x'(t)$) depends only on the difference $t - s$, and not on the specific values of t and s. In some cases this assumption is too restrictive; more realistic models often call for time-dependent feedback. The equation (1.2) should then be replaced by

$$x(t) + \int_0^t k(t,s)x(s)\,\mathrm{d}s = f(t), \quad t \geq 0. \tag{1.5}$$

A similar change is required in (1.4). Equation (1.5) is a nonconvolution Volterra equation of the second kind.

In the equations above the feedback is assumed to be linear; normally this is an acceptable first approximation. However, in a more detailed analysis one must often take nonlinearities into account. For example, the state at time t may depend on $x(s)$ for $s < t$ through a nonlinear function g (and through the feedback kernel k). If both the nonlinearity and the feedback are taken to be time invariant, then we end up with the nonlinear integral equation

$$x(t) + \int_0^t k(t-s)g(x(s))\,\mathrm{d}s = f(t), \quad t \geq 0,$$

as the description of our system. To complicate things further, we may have a nonlinear, time-dependent feedback, we may have time-dependent nonlinearities, etc. Note that one need not always have the nonlinearity inside the integral. If, for example, one takes $y(t) = g(x(t))$ above, then y satisfies the equation

$$y(t) = g\left(f(t) - \int_0^t k(t-s)y(s)\,\mathrm{d}s\right), \quad t \geq 0.$$

A feature common to all of the integral or integrodifferential equations above is the fact that the supports of the kernels k and μ are such that the independent variable t appears as the upper limit of integration. This is, of course, only another way of stating that the present depends on the past, but not on the future.

Equations involving integrals where the integration does not extend past the independent variable take their name from Vito Volterra, who was one of the first to investigate equations of this type. (For a short biography of Volterra, 1860–1940, see Volterra [4]. This reference also contains a complete list of Volterra's publications.) The class of equations which one gets by dropping this requirement on the upper limit of integration, i.e., equations of the form

$$x(t) + \int_0^b k(t,s)x(s)\,\mathrm{d}s = f(t), \quad 0 \leq t \leq b,$$

are called integral equations of Fredholm type. Thus, formally a Volterra equation is a special case of a Fredholm equation. However, much of the classical theory of Fredholm equations reduces to mere trivialities when applied to Volterra equations. On the other hand, Volterra equations exhibit a variety of phenomena unknown to Fredholm theory. In addition, it is generally true that Fredholm equations describe boundary value problems and are related to operator theory, whereas Volterra equations describe initial value problems and are related to dynamical systems. Hence there is every reason to consider these fields separately; the fact that integrals appear in both types of equations should not make one overlook the basic differences. This book concentrates on the Volterra theory, but those results that are common to Volterra and Fredholm equations are formulated in such generality that they can be applied to Fredholm equations as well. This is, in particular, true for equations that are required to hold on the whole real line $(-\infty, \infty)$.

The overwhelming majority of the integral equations we analyse are of the second kind. Only in Chapter 5 do we treat equations of the first kind in more detail. These are equations where the unknown function only appears under the integral sign. Thus,

$$\int_0^t k(t-s)x(s)\,\mathrm{d}s = f(t), \quad t > 0,$$

is a convolution Volterra equation of the first kind. Analogously,

$$\int_0^t k(t,s)x(s)\,\mathrm{d}s = f(t), \quad t > 0,$$

is a nonconvolution Volterra equation of the first kind. Trivially, if the kernel k and the nonhomogeneous term f are sufficiently smooth and k satisfies $0 < |k(0)| < \infty$, then the former equation may be reduced to an equation of the second kind by differentiation. A similar comment applies to the nonconvolution equation.

2. Some Examples of Volterra Equations

Volterra equations appear in a variety of applied problems. Below we give a few examples.

2.1 Example

A model for cosmic ray transport was presented in Klimas and Sandri [1] and subjected to further investigation in Hanson *et al.* [1] and [2]. It is shown that the transport of charged particles in a turbulent plasma, e.g., the cosmic rays in the interplanetary solar wind, is described by the coupled equations

$$\begin{aligned} &\frac{\partial I}{\partial \tau} + \alpha \nabla \cdot \Phi = 0, \\ &\frac{\partial \Phi}{\partial \tau} + \epsilon \Omega \cdot \Phi = -(\epsilon\eta)^2 \int_0^\tau \mathcal{K}(\epsilon, \lambda) \cdot \Phi(\tau - \lambda)\,\mathrm{d}\lambda - \tfrac{1}{3}\alpha \nabla I. \end{aligned} \tag{2.1}$$

Here I is the omnidirectional intensity, Φ is the flux, and the parameters $(\epsilon\eta)^2$ and α are small. The term $\frac{1}{3}\alpha\nabla I$ is a source term due to gradients in the density. The tensor kernel $\mathcal{K}(\epsilon, \lambda)$ is computed from the power spectrum associated with the random magnetic field, and decays as $\frac{1}{\lambda}$ for λ large. The fact that the kernel is nonintegrable over $\mathbf{R}^+$ causes the adiabatic approximation to be invalid. (See Klimas and Sandri [1] for details.)

For (2.1), the scalar approximation

$$\frac{\partial}{\partial t}F(t,\alpha)+\alpha\int_0^t k(t-s)F(s,\alpha)\,\mathrm{d}s=-\sigma(t),\quad t>0;\quad F(0,\alpha)=1,\ (2.2)$$

was presented in Hanson *et al.* [2]. The kernel $k(t)$ is assumed to satisfy $k(t)=(1+t)^{-1}$; α plays the role of $(\epsilon\eta)^2$ in (2.1). The nonhomogeneous term $\sigma(t)$ represents the density gradient. The goal is to acquire information on the behaviour of the flux $F(t,\alpha)$, in particular as $t\to\infty$ and $\alpha\downarrow 0$.

In Hanson *et al.* [1], a detailed analysis of (2.2) is carried out using Laplace transforms. Several results on how $F(t,\alpha)$ behaves for small α are obtained. See also Hanson *et al.* [2] for numerical work, and Ling [1] for some qualitative extensions. Here we observe that, by Theorem **3**.3.3, one has $F(t,\alpha)=r_\alpha(t)-(r_\alpha*\sigma)(t)$ where r_α is the differential resolvent of $\alpha/(1+t)$. Since the kernel is completely monotone, one may apply Theorem **5**.4.1. This leads to the conclusion that for every $\alpha>0$ the function r_α is both bounded and integrable on $\mathbf{R}^+$. Consequently, if σ is either bounded or integrable on $\mathbf{R}^+$, then one has $\sup_{t\in\mathbf{R}^+}|F(t,\alpha)|<\infty$. If we, in addition, employ Exercise **6**.13, and assume that σ is integrable, then we conclude that this bound is uniform with respect to α. Moreover, one can show that

$$r_\alpha(t)=\left(\alpha t\ln^2(t)\right)^{-1}+O\left(\left(t\ln^3(t)\right)^{-1}\right),$$

as $t\to\infty$. See Wong and Wong [1], Theorem 4. Further results on how $F(t,\alpha)$ and r_α depend on α may be obtained by consulting, e.g., Hannsgen [11] and [15]. See also Jordan and Wheeler [6], p. 106.

2.2 Example

Population dynamics constitutes a major source of Volterra equations, the classical reference being Volterra [5]. Let us briefly sketch one example from this field.

Consider a population having an age distribution $y(t,a)$, $t\geq 0$, $a\geq 0$. Thus $\int_A y(t,a)\,\mathrm{d}a$ is the number of individuals with age in the set A at time t. Assume that the process of aging and dying is modelled by the balance law

$$\frac{\partial y(t,a)}{\partial t}+\frac{\partial y(t,a)}{\partial a}=-m(a)y(t,a).\qquad(2.3)$$

The nonnegative function m denotes the age-dependent death rate. Suppose that the birth process satisfies

$$y(t,0)=\int_{\mathbf{R}^+}y(t,a)b(a)\,\mathrm{d}a,\qquad(2.4)$$

where b is the age-dependent fertility. Finally assume that the initial age distribution $y(0,a)=\phi(a)$ is known.

If one solves (2.3) by the method of characteristics and makes use of the initial condition, then one obtains

$$y(t,a) = \begin{cases} \phi(a-t)\exp\left(-\int_0^t m(s+a-t)\,\mathrm{d}s\right), & 0 \le t < a, \\ y(t-a,0)\exp\left(-\int_0^a m(s)\,\mathrm{d}s\right), & t \ge a. \end{cases} \tag{2.5}$$

Inserting the expression for $y(t,a)$ given by (2.5) into (2.4), we obtain the linear Volterra equation

$$y(t,0) + \int_0^t k(t-s)y(s,0)\,\mathrm{d}s = f(t), \quad t \ge 0, \tag{2.6}$$

where

$$k(t) = -b(t)\exp\left(-\int_0^t m(s)\,\mathrm{d}s\right),$$

$$f(t) = \int_{\mathbf{R}^+} \phi(s)\exp\left(-\int_0^t m(s+\sigma)\,\mathrm{d}\sigma\right) b(t+s)\,\mathrm{d}s.$$

Equation (2.6) is the classical renewal equation; see Feller [1].

A more realistic model is obtained if one includes density-dependence as well. Then

$$\begin{aligned} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} &= -m\big(a, N(t)\big)y(t,a), \\ y(t,0) &= \int_{\mathbf{R}^+} y(t,a)b\big(a, N(t)\big)\,\mathrm{d}a, \\ y(0,a) &= \phi(a), \\ N(t) &= \int_{\mathbf{R}^+} y(t,a)\,\mathrm{d}a. \end{aligned}$$

This model was analysed in Gurtin and MacCamy [1] and has been extensively developed. For notes on this development, and for a comprehensive account of age-dependent population dynamics in general (including a wealth of references), see Webb [1]. A detailed account of mathematical models for physiologically structured populations can be found in Metz and Diekmann [1].

For a general treatment of age-dependent processes and renewal theory, see Chapter IV of Athreya and Ney [1].

2.3 Example

The spread of an epidemic in a population of fixed size can be mathematically described as follows (Diekmann [2]). Assume that the population consists of infected ($I(t)$), and susceptible ($S(t)$) individuals with $I(t)+S(t) = P$. Further assume that transitions from the infected state to the susceptible state cannot occur. Then $I'(t)$ is the rate at which susceptible individuals become infected. We take the infected population $I(t)$ to be structured according to the time since infection; i.e., $i(t,s)\,\mathrm{d}s$ denotes the number of individuals that were infected between $t-s$ and $t-s+\mathrm{d}s$. Then

$$I(t)=\int_{\mathbf{R}^+} i(t,s)\,\mathrm{d}s,\quad t\ge 0,$$

and $\frac{\partial i}{\partial t}+\frac{\partial i}{\partial s}=0$. Hence,

$$\begin{aligned} i(t,s)&=i(t-s,0),\quad t>s,\\ i(t,s)&=i(0,s-t),\quad t<s.\end{aligned}$$

Moreover,

$$\frac{\mathrm{d}S}{\mathrm{d}t}=-\frac{\mathrm{d}I}{\mathrm{d}t}=-i(t,0),\quad t>0.$$

Suppose that there exists a nonnegative function A such that the infectivity B (the rate at which susceptibles become infected) is given by

$$B(t)=\int_{\mathbf{R}^+} i(t,s)A(s)\,\mathrm{d}s.$$

Then

$$\frac{\mathrm{d}S}{\mathrm{d}t}=-S(t)\int_{\mathbf{R}^+} i(t,s)A(s)\,\mathrm{d}s,\quad t>0,$$

with $S(0)=S_0>0$ given. Divide by $S(t)$ and integrate over $(0,t)$ to obtain

$$\ln\left(\frac{S(t)}{S_0}\right)=\int_0^t \big(S(t-s)-S_0\big)A(s)\,\mathrm{d}s-f(t),$$

where

$$f(t)=\int_0^t\int_{\mathbf{R}^+} i(0,\tau)A(s+\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

If we let $x(t)=-\ln(S(t)/S_0)$, then we end up with the nonlinear Volterra equation

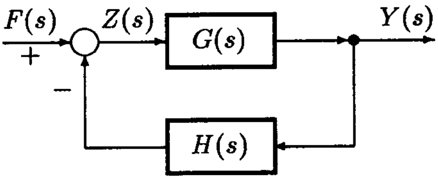
$$x(t)=S_0\int_0^t A(t-s)g(x(s))\,\mathrm{d}s+f(t),\quad t\ge 0, \tag{2.7}$$

where $g(x)=1-\mathrm{e}^{-x}$.

The classical paper on this model is Kermack and McKendrick [1]. An asymptotic analysis of the solutions of (2.7) was done in Diekmann [1], in Diekmann and Kaper [1], and in Gripenberg [27]. Space-dependence was incorporated in Diekmann [2]. For models including susceptible, infected, and removed individuals, see, e.g., Stech and Williams [1]. In Gripenberg [20], the spread of an infectious disease that does not induce permanent immunity is studied.

For a survey of epidemics in homogeneous populations, see Hethcote, Stech, and Van den Driessche [1]. Further work on mathematical epidemiology is to be found in Capasso, Grosso, and Paveri-Fontana [1], pp. 106–189 and in Hethcote, Lewis, and Van den Driessche [1].

2.4 Example
Volterra equations have frequently been used to describe control systems with feedback loops. We illustrate by the single closed-loop, time invariant, and linear system shown below:



Here $F(s)$ and $Y(s)$ are, respectively, the transforms of the input $f(t)$ and the output $y(t)$. The system transfer function $G(s)$ is defined as the ratio of the transform of the output $Y(s)$ to the transform of the input $Z(s)$. The feedback transfer function is denoted by $H(s)$. Consequently,

$$Y(s) = G(s)\Big[F(s) - Y(s)H(s)\Big],$$

or

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} F(s).$$

Let g and h be the inverse transforms of G and H, and write $k = g * h$, $f_1 = g * f$. We then have, in the time domain,

$$y(t) + \int_0^t k(t-s)y(s)\,\mathrm{d}s = f_1(t), \quad t \geq 0.$$

If the feedback is built up by a nonlinear term N with no memory and a linear time invariant part with memory, then the resulting equation is

$$y(t) + \int_0^t k(t-s)N\big(y(s)\big)\,\mathrm{d}s = f_1(t), \quad t \geq 0.$$

In case the system is time varying, we are led to a nonconvolution equation. Of course, a stable system is the foremost goal. In particular one wishes to have input–output stability in an L^p-sense, frequently for $p = \infty$ or $p = 2$.

The literature in the field is huge; see MacFarlane [1] for a review and reprints of important papers. For more recent work, see, e.g., Mousa, Miller, and Michel [1] and the references mentioned therein.

2.5 Example
The point model equations for a nuclear reactor with linear feedback may be written as

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}p(t) &= \frac{k(t)-\beta}{l}p(t) + \sum_{i=1}^{m} \lambda_i c_i(t), \\ \tfrac{\mathrm{d}}{\mathrm{d}t}c_i(t) &= \frac{\beta_i}{l}p(t) - \lambda_i c_i(t), \quad i = 1, \dots, m, \\ k(t) &= -\int_0^t K(t-s)\big(p(s) - p_0\big)\,\mathrm{d}s + f(t), \quad t \geq 0. \end{aligned}\right\} \tag{2.8}$$

Above p denotes the neutron density, k is the reactivity, β_i, $i = 1, \dots, m$, are the delayed neutron fractions ($\beta = \sum_{i=1}^{m} \beta_i$), l is the neutron average lifetime, and λ_i are the decay constants of the delayed neutrons. The functions c_i represent the delayed neutron densities, K is the reactivity feedback kernel, p_0 denotes the equilibrium power level, and f is the external control. It is obviously of primary interest to have stability of the equilibrium solution $p = p_0$, $c_i = \beta_i p_0/(l\lambda_i)$.

A large number of results on stability criteria for (2.8) exist. See Londen [1], [3], and [4], Podowski [1] and [2], and the references mentioned in these articles. General treatises on nuclear reactor dynamics are, for example, Ash [1] and Akcasu, Lellouche, and Shotkin [1].

Observe that if the delayed neutrons are neglected then (2.8) reduces to

$$\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t)\right)}{p(t)} + \frac{1}{l}\int_0^t K(t-s)\big(p(s) - p_0\big)\,\mathrm{d}s = \frac{f(t)}{l}.$$

Define $x(t) = \ln(p(t)/p_0)$, $a(t) = p_0 K(t)/l$, and $h(t) = f(t)/l$. Then

$$x'(t) + \int_0^t a(t-s)(\mathrm{e}^{x(s)} - 1)\,\mathrm{d}s = h(t).$$

We end up with a nonlinear Volterra integrodifferential equation having a nonlinearity which is *a priori* bounded from below.

More complicated models may be constructed, for example, with interacting cores or including space-dependence.

2.6 Example

The theory of superfluidity has motivated several authors to study the following problem.

Let $(x, y, z) \in \mathbf{R}^3$ and let two infinite plates be located at $x = 0$ and $x = L$, respectively. Assume that the region between the plates contains liquid helium. Suppose that the boundary plates are given sinusoidal oscillations in the y-direction. This will create a one-dimensional flow in the liquid. Denote the velocity profile at the point (t, x, y, z) by $u(t, x)$. Then u will be governed by the partial differential equation

$$u_t = u_{xx}, \quad t > 0, \quad 0 < x < L,$$

subject to the conditions

$$\begin{aligned} u(0,x) &= u_0(x), \quad 0 < x < L, \\ u_x(t,0) &= c_1\big(u(t,0) - c_2\sin(k_1 t)\big)^3, \quad t > 0, \\ u_x(t,L) &= -c_1\big(u(t,L) - c_2\sin(k_2 t)\big)^3, \quad t > 0, \end{aligned}$$

where c_1, c_2, k_1, k_2 are positive constants, and u_0 is the initial velocity profile. This problem may be converted to the system of nonlinear Volterra equations

$$x_1(t) + \int_0^t a_1(t-s)g_1\big(s, x_1(s)\big)\,\mathrm{d}s + \int_0^t a_2(t-s)g_2\big(s, x_2(s)\big)\,\mathrm{d}s = f_1(t),$$
$$x_2(t) + \int_0^t a_2(t-s)g_1\big(s, x_1(s)\big)\,\mathrm{d}s + \int_0^t a_1(t-s)g_2\big(s, x_2(s)\big)\,\mathrm{d}s = f_2(t),$$

where $x_1(t) = u(t, 0)$, $x_2(t) = u(t, L)$, and

$$g_i(t, u) = c_1\big(u - c_2\sin(k_i t)\big)^3, \quad i = 1, 2.$$

Clearly, these functions g_i are periodic in t. The kernels a_1 and a_2 are defined by

$$a_1(t) = \frac{1}{L}\Big(1 + 2\sum_{n=1}^{\infty}\exp\big(-(n\pi/L)^2 t\big)\Big),$$

and

$$a_2(t) = \frac{1}{L}\Big(1 + 2\sum_{n=1}^{\infty}(-1)^n\exp\big(-(n\pi/L)^2 t\big)\Big).$$

The nonhomogeneous terms $f_1(t)$ and $f_2(t)$ depend on the initial function u_0. These two terms are asymptotically almost-periodic.

For this problem, one can show that solutions exist for all $t \ge 0$ and are asymptotically almost-periodic. See Miller [7], [8], and [12], and Gripenberg [21], for this result and extensions.

In Levinson [1], the corresponding semi-infinite problem is considered, i.e., the equation

$$u_t = u_{xx}, \quad x > 0, \quad t > 0,$$

subject to

$$u(x, 0) = 0, \quad x > 0; \quad u_x(0, t) = \Phi\big(u(0, t) - f(t)\big), \quad t > 0.$$

Here Φ is monotone increasing, locally Lipschitz and satisfies $\Phi(0) = 0$ and $f(t)$ is periodic. The problem is converted to

$$u(0, t) + \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-\frac{1}{2}}\Phi\big(u(0, s) - f(s)\big)\,\mathrm{d}s = 0, \quad t \ge 0,$$

and conditions are given under which $\lim_{t\to\infty}(u(0, t) - \varphi(t)) = 0$, where φ is periodic. Note that from $u(0, t)$ on may easily compute $u(x, t)$.

2.7 Example

Volterra equations are common in mathematical viscoelasticity pertaining to materials with memory. These materials are characterized by constitutive relations which are functionals of the past history of the material. In this context, we examine only a simple example and refer the reader to Renardy, Hrusa, and Nohel [1] for an in-depth overview of the field. (In particular, see Section IV 4 of this monograph.) For a brief, easily accessible account of initial value, integrodifferential equations in viscoelasticity, see Hrusa, Nohel, and Renardy [1].