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Center Manifolds

The main goal of this book is to study some bifurcation phenomena of vector fields. This is, in general, a complicated problem. As a preliminary step, it is necessary to simplify the problem as much as possible without changing the dynamic behavior of the original vector field. There are two steps for this purpose: to reduce the dimension of the bifurcation problem by using the center-manifold theory, which will be introduced in this chapter, and to make the equation as simple as possible by using normal-form theory which will be discussed in the next chapter.

We first give some rough ideas about center manifolds. Consider a differential equation

$$\dot{x} = Ax + f(x), \qquad (\mathrm{A})_f$$

where $x \in \mathbb{R}^n$, $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ for some $k \geq 1$, $f(0) = 0$, and $Df(0) = 0$.

We write the spectrum $\sigma(A)$ of A as

$$\sigma(A) = \sigma_s \cup \sigma_c \cup \sigma_u,$$

where

$$\sigma_s = \{\lambda \in \sigma(A) \,|\, \mathrm{Re}\,\lambda < 0\},$$

$$\sigma_c = \{\lambda \in \sigma(A) \,|\, \mathrm{Re}\,\lambda = 0\},$$

$$\sigma_u = \{\lambda \in \sigma(A) \,|\, \mathrm{Re}\,\lambda > 0\}.$$

Let E_s, E_c, and E_u be the generalized eigenspaces corresponding to σ_s,

σ_c, and σ_u, respectively. Then we have

$$\mathbb{R}^n = E_s \oplus E_c \oplus E_u,$$

with corresponding projections

$$\pi_s: \mathbb{R}^n \to E_s, \qquad \pi_c: \mathbb{R}^n \to E_c, \qquad \pi_u: \mathbb{R}^n \to E_u.$$

It is well known that if A is hyperbolic, that is, $\sigma_c = \varnothing$, then the flow of $(\mathrm{A})_f$ in a small neighborhood Ω of the equilibrium point $x = 0$ is topologically equivalent to the flow of the linearized equation at $x = 0$

$$\dot{x} = Ax. \tag{(A)$_0$}$$

Since $x(t) = e^{At}x(0)$ is the solution of $(\mathrm{A})_0$ and $\sigma_c = \varnothing$, any nonzero solution in E_s (or E_u) tends to the equilibrium $x = 0$ exponentially as $t \to +\infty$ (or $t \to -\infty$). Therefore, the structure of flow in Ω is simple; it is also stable with respect to any small perturbation on the right-hand side of equation $(\mathrm{A})_0$. See Hartman [1], for example.

However, if $\sigma_c \neq \varnothing$, then the situation will be different from the above in two aspects. First, the topological structure for $(\mathrm{A})_f$ is not, in general, the same as for $(\mathrm{A})_0$ any more; this will be shown in a lot of examples in Chapters 3–5. Second, more complicated structure of the flow for $(\mathrm{A})_f$ may exist on an invariant manifold $W^c(f)$, and the dimension of $W^c(f)$ is equal to the dimension of E_c.

In fact, if $f \equiv 0$, then all bounded solutions of $(\mathrm{A})_0$, including all equilibria and periodic orbits, are contained in the subspace E_c, which is invariant under $(\mathrm{A})_0$. So we take $W^c(0) = E_c$. We will prove that the aforementioned $W^c(f)$ exists for $f \not\equiv 0$, it is tangent to E_c at $x = 0$, and $W^c(f)$ contains all solutions of $(\mathrm{A})_f$ that stay in Ω for all $t \in \mathbb{R}^1$. In particular, $W^c(f)$ contains all sufficiently small equilibria, periodic orbits, and homoclinic and heteroclinic orbits. And if $\sigma_u = \varnothing$, then all solutions of $(\mathrm{A})_f$ (in Ω) will converge exponentially to some solutions on $W^c(f)$ as $t \to +\infty$. Therefore, instead of the n-dimensional equation $(\mathrm{A})_f$, we can consider a lower-dimensional equation on $W^c(f)$ for a bifurcation problem, and $W^c(f)$ is called a center manifold. The precise definition will be given subsequently in Section 1.1.

We will prove the existence, uniqueness, and smoothness of global center manifolds in Sections 1.1–1.2 under a quite strong condition which says the Lipschitz constant of f is globally small. In Section 1.3 the cut-off technique is used to get the local center manifolds from the global theory, and the above Lipschitz condition will be satisfied automatically since $f(0) = 0$ and $Df(0) = 0$. But a new problem arises: The local center manifold is not unique. In fact, different cut-off functions can give different local center manifolds. Hence, it is needed to show the equivalence (in some sense) between different local center manifolds concerning the bifurcation problems. Finally, in Section 1.4 we discuss the center-stable and center-unstable manifolds, give the asymptotic behavior of any solution of (1.1) in $\mathbb{R}^n$, and describe the invariant foliation structure.

1.1 Existence and Uniqueness of Global Center Manifolds

Consider the equation

$$\dot{x} = Ax + f(x), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ for some $k \geq 1$, $f(0) = 0$, and $Df(0) = 0$.

We keep the notations E_s, E_c, E_u and π_s, π_c, π_u throughout this chapter, and let

$$E_h = E_s \oplus E_u, \qquad \pi_h = \pi_s + \pi_u.$$

As usual, we denote by $|y|$ the norm of y in some Banach space. Let X, Y be Banach spaces and $C^k(X,Y)$ be the set of all C^k mappings from X into Y. We define the Banach space

$$C_b^k(X,Y) = \left\{ w \in C^k(X,Y) \,\middle|\, \|w\|_{C^k} := \max_{0 \leq j \leq k} \sup_{x \in X} |D^j w(x)| < \infty \right\}.$$

If $X = Y$, we write $C_b^k(X,X)$ as $C_b^k(X)$. We let

$$\|Dw\| = \sup_{x \in X} |Dw(x)|.$$

Similarly, we define

$$C_b^{k,1}(X,Y) = \left\{ w \in C_b^k(X,Y) \,\middle|\, \sup \frac{\|D^k w(x) - D^k w(y)\|}{\|x - y\|_X} < \infty, \right.$$
$$\left. x, y \in X, x \neq y \right\}$$

with norm

$$\|w\|_{C^{k,1}} = \|w\|_{C^k} + \sup \frac{\|D^k w(x) - D^k w(x)\|}{\|x-y\|_X}, \qquad x, y \in X,\ x \neq y.$$

Finally, we denote by $\tilde{x}(t,x)$ the solution of (1.1) with the initial condition $\tilde{x}(0,x) = x$.

Now we state the main result of this section, and will prove it by using several lemmas.

Theorem 1.1. (i) *There is a positive number* δ_0 *which depends only on* A *in (1.1) such that if* $f \in C_b^{0,1}(\mathbb{R}^n)$ *and* $\mathrm{Lip}(f) < \delta_0$, *then the set*

$$W^c := \left\{ x \in \mathbb{R}^n \,\middle|\, \sup_{t\in\mathbb{R}} |\pi_h \tilde{x}(t,x)| < \infty \right\} \tag{1.2}$$

is invariant under (1.1) and is a Lipschitz submanifold of $\mathbb{R}^n$; *more precisely, there exists a unique Lipschitz function* $\psi \in C_b^0(E_c, E_h)$ *such that*

$$W^c = \{x_c + \psi(x_c) \mid x_c \in E_c\}. \tag{1.3}$$

(ii) *If* $\phi \in C_b^0(E_c, E_h)$, *and the set*

$$M_\phi := \{x_c + \phi(x_c) \mid x_c \in E_c\} \tag{1.4}$$

is invariant under (1.1), *then* $M_\phi = W^c$ *and* $\phi = \psi$.

Definition 1.2. W^c is called the global center manifold of (1.1).

Remark 1.3. If $f \in C_b^1(\mathbb{R}^n)$, then we will usually replace the condition $\mathrm{Lip}(f) < \delta_0$ by $\|Df\| < \delta_0$.

Remark 1.4. The uniqueness conclusion (ii) should be understood in the following sense: If M_ϕ is invariant under (1.1), then $\phi \in C_b^0(E_c, E_h)$ is determined uniquely. This is not true if we replace the condition

$\phi \in C_b^0(E_c, E_h)$ by $\phi \in C^0(E_c, E_h)$ unless $A|_{E_c}$ is semisimple and f has compact support (see Sijbrand [1] and Vanderbauwhede [3] for more details).

Lemma 1.5. *For any integer $k > 0$, there are constants $K \geq 1$, $\alpha > 0$, and $\beta > 0$ such that $k\alpha < \beta$, and*

$$\begin{aligned} |e^{At}\pi_c| &\leq Ke^{\alpha|t|}, & t &\in \mathbb{R}, \\ |e^{At}\pi_s| &\leq Ke^{-\beta t}, & t &\geq 0, \\ |e^{At}\pi_u| &\leq Ke^{\beta t}, & t &\leq 0. \end{aligned} \tag{1.5}$$

Proof. Let

$$\beta = \min\{|\mathrm{Re}\,\lambda| \,|\, \lambda \in \sigma_u \cup \sigma_s|\} - \epsilon,$$

$$0 < \epsilon < \alpha < \frac{\beta}{k},$$

where ϵ and α are sufficiently small. Thus, the existence of K is obvious by the properties of e^{At}. □

Let γ satisfy

$$\alpha < \gamma < \beta. \tag{1.6}$$

Define a Banach space by

$$C_\gamma := \left\{ x \in C^0(\mathbb{R}, \mathbb{R}^n) \,\middle|\, \|x\|_\gamma := \sup_{t \in \mathbb{R}} e^{-\gamma|t|}|x(t)| < \infty \right\}.$$

The following lemma gives a different criterion for W^c.

Lemma 1.6. *Suppose $f \in C_b^{0,1}(\mathbb{R}^n)$ and (1.6) is satisfied. Then*
(i)

$$W^c = \{x \in \mathbb{R}^n \mid \tilde{x}(\cdot, x) \in C_\gamma\}. \tag{1.7}$$

(ii) *Consider the integral equation*

$$y(t) = e^{At}\pi_c x + \int_0^t e^{A(t-\tau)}\pi_c f(y(\tau))\,d\tau$$

$$+ \int_\infty^t e^{A(t-\tau)}\pi_u f(y(\tau))\,d\tau + \int_{-\infty}^t e^{A(t-\tau)}\pi_s f(y(\tau))\,d\tau. \quad (1.8)$$

We have

$$W^c = \{y(0) \in \mathbb{R}^n \mid y(\cdot) \in C_\gamma \text{ and satisfies (1.8) for some } x \in \mathbb{R}^n\}. \quad (1.9)$$

Proof. By the variation of constants formula, for $t_0, t \in \mathbb{R}$ we have

$$\tilde{x}(t,x) = e^{A(t-t_0)}\tilde{x}(t_0,x) + \int_{t_0}^t e^{A(t-\tau)} f(\tilde{x}(\tau,x))\,d\tau. \quad (1.10)$$

Denote by $\tilde{W}^c$ the right-hand side of (1.7), and by $\tilde{\tilde{W}}^c$ the right-hand side of (1.9). We will show that $W^c \subset \tilde{W}^c \subset \tilde{\tilde{W}}^c \subset W^c$.

(a) Suppose $x \in W^c$; then by (1.2)

$$\sup_{t\in\mathbb{R}} e^{-\gamma|t|}|\pi_h \tilde{x}(t,x)| \le \sup_{t\in\mathbb{R}}|\pi_h \tilde{x}(t,x)| < \infty. \quad (1.11)$$

Taking $t_0 = 0$ in (1.10) we obtain

$$\pi_c \tilde{x}(t,x) = e^{At}\pi_c x + \int_0^t e^{A(t-\tau)}\pi_c f(\tilde{x}(\tau,x))\,d\tau. \quad (1.12)$$

Using Lemma 1.5 and (1.6), we have from (1.12) that

$$|\pi_c \tilde{x}(t,x)| \le Ke^{\gamma|t|}|x| + K\|f\|_{C^0}\left|\int_0^t e^{\gamma(t-\tau)}d\tau\right| \le Ke^{\gamma|t|}\left(|x| + \frac{\|f\|_{C^0}}{\gamma}\right),$$

whence

$$\sup_{t\in\mathbb{R}} e^{-\gamma|t|}|\pi_c \tilde{x}(t,x)| < \infty. \quad (1.13)$$

It follows from (1.11) and (1.13) that $x \in \tilde{W}^c$, and this implies $W^c \subset \tilde{W}^c$.

(b) Suppose now $x \in \tilde{W}^c$, that is, $\tilde{x}(\cdot, x) \in C_\gamma$. From (1.10) we have

$$\pi_u \tilde{x}(t,x) = e^{A(t-t_0)}\pi_u \tilde{x}(t_0,x) + \int_{t_0}^{t} e^{A(t-\tau)}\pi_u f(\tilde{x}(\tau,x))\,d\tau. \tag{1.14}$$

Fixing $t \in \mathbb{R}$ and $t_0 \geq \max(t,0)$, we obtain from (1.5)

$$\begin{aligned}|e^{A(t-t_0)}\pi_u \tilde{x}(t_0,x)| &\leq Ke^{\beta(t-t_0)}|\tilde{x}(t_0,x)| \\ &\leq Ke^{\beta t-(\beta-\gamma)t_0}\|\tilde{x}(\cdot,x)\|_\gamma \to 0 \quad \text{as } t_0 \to +\infty,\end{aligned}$$

since $\tilde{x}(\cdot, x) \in C_\gamma$ and $\gamma < \beta$. Therefore, taking the limit in (1.14) as $t_0 \to +\infty$, we have

$$\pi_u \tilde{x}(t,x) = \int_{-\infty}^{t} e^{A(t-\tau)}\pi_u f(\tilde{x}(\tau,x))\,d\tau. \tag{1.15}$$

Similarly, we can obtain

$$\pi_s \tilde{x}(t,x) = \int_{-\infty}^{t} e^{A(t-\tau)}\pi_s f(\tilde{x}(\tau,x))\,d\tau. \tag{1.16}$$

Combining (1.12), (1.15), and (1.16), we see that $\tilde{x}(\cdot, x)$ satisfies (1.8). Thus $x = \tilde{x}(0,x) \in \widetilde{\widetilde{W}}{}^c$. Therefore $\tilde{W}^c \subset \widetilde{\widetilde{W}}{}^c$.

(c) Suppose $y_0 \in \widetilde{\widetilde{W}}{}^c$, that is, there is a function $y(\cdot) \in C_\gamma$, which satisfies (1.8) for some $x \in \mathbb{R}^n$ and $y(0) = y_0$. Then from (1.8)

$$\begin{aligned}y(t) &= e^{At}\left\{\pi_c x + \int_{-\infty}^{0} e^{-A\tau}\pi_s f(y(\tau))\,d\tau + \int_{\infty}^{0} e^{-A\tau}\pi_u f(y(\tau))\,d\tau\right\} \\ &\quad + \int_0^t e^{A(t-\tau)} f(y(\tau))\,d\tau \\ &= e^{At}y_0 + \int_0^t e^{A(t-\tau)} f(y(\tau))\,d\tau.\end{aligned}$$

Hence $y(t)$ is the solution of (1.1) with initial value $y(0) = y_0$. Using (1.5) and (1.8), it follows that

$$|\pi_u y(t)| \leq K\|f\|_{C^0}\int_t^{\infty} e^{\beta(t-\tau)}\,d\tau = \frac{K}{\beta}\|f\|_{C^0} < \infty,$$

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and

$$|\pi_s y(t)| \leq \frac{K}{\beta} \|f\|_{C^0} < \infty,$$

since $f \in C_b^0(\mathbb{R}^n)$. Hence $|\pi_h y(t)| < \infty$. Thus $y_0 \in W^c$. This implies $\widetilde{\widetilde{W}}^c \subset W^c$. □

Now we consider the integral equation defined by (1.8). Let F: $E_c \to C_\gamma$ be defined by

$$F(\xi)(t) = e^{At}\xi, \qquad \xi \in E_c, \tag{1.17}$$

and G: $C_\gamma \to C_\gamma$ be defined by

$$G(y(\cdot))(t) = \int_0^t e^{A(t-\tau)}\pi_c y(\tau)\,d\tau + \int_\infty^t e^{A(t-\tau)}\pi_u y(\tau)\,d\tau + \int_{-\infty}^t e^{A(t-\tau)}\pi_s y(\tau)\,d\tau. \tag{1.18}$$

We denote the previous three integrals by $G_c(y(\cdot))(t)$, $G_u(y(\cdot))(t)$ and $G_s(y(\cdot))(t)$, respectively. We will use these notations repeatedly in this chapter.

Define J: $E_c \times C_\gamma \to C_\gamma$ by

$$J(\xi, y) = F(\xi) + G(f(y(\cdot))). \tag{1.19}$$

Obviously, if $\xi \in E_c$ then $y = y^*(\cdot)$ is a fixed point of $J(\xi, \cdot)$ if and only if $y^*(t)$ is a solution of (1.8) with $x = \xi$.

Lemma 1.7. *There is a number $\delta_0 > 0$, which depends only on A, such that, if $\mathrm{Lip}(f) < \delta_0$, then for any $\xi \in E_c$, $J(\xi, y)$, defined by (1.19), has a unique fixed point $y = x^*(\cdot, \xi)$.*

Proof. Note that

$$\begin{aligned} J(\xi, y_1) - J(\xi, y_2) &= G(f(y_1(\cdot))) - G(f(y_2(\cdot))) \\ &= G(f(y_1(\cdot)) - f(y_2(\cdot))), \end{aligned} \tag{1.20}$$

and by (1.5) we have

$$
\begin{aligned}
&\left|G_c(f(y_1(\cdot)) - f(y_2(\cdot)))(t)\right| \\
&\quad = \left|\int_0^t e^{A(t-\tau)}\pi_c(f(y_1(\tau)) - f(y_2(\tau)))d\tau\right| \\
&\quad \le K\,\mathrm{Lip}(f)\left|\int_0^t e^{\alpha|t-\tau|}|y_1(\tau) - y_2(\tau)|d\tau\right| \\
&\quad \le K\,\mathrm{Lip}(f)\left|\int_0^t e^{\alpha|t-\tau|}e^{\gamma|\tau|}\left(\sup_{\tau\in\mathbb{R}} e^{-\gamma|\tau|}|y_1(\tau) - y_2(\tau)|\right)d\tau\right| \\
&\quad \le \frac{e^{\gamma|t|}}{\gamma-\alpha}K\,\mathrm{Lip}(f)\|y_1 - y_2\|_\gamma.
\end{aligned}
\tag{1.21}
$$

Similarly, we have

$$
|G_u(f(y_1(\cdot)))(t) - G_u(f(y_2(\cdot)))(t)| \le \frac{e^{\gamma|t|}}{\beta-\gamma}K\,\mathrm{Lip}(f)\|y_1 - y_2\|_\gamma,
\tag{1.22}
$$

$$
|G_s(f(y_1(\cdot)))(t) - G_s(f(y_2(\cdot)))(t)| \le \frac{e^{\gamma|t|}}{\beta-\gamma}K\,\mathrm{Lip}(f)\|y_1 - y_2\|_\gamma.
\tag{1.23}
$$

These estimates give

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} e^{-\gamma|t|}|G(f(y_1(\cdot)))(t) - G(f(y_2(\cdot)))(t)| \\
&\quad \le K\left(\frac{1}{\gamma-\alpha} + \frac{2}{\beta-\gamma}\right)\mathrm{Lip}(f)\|y_1 - y_2\|_\gamma.
\end{aligned}
$$

We choose

$$
\delta_0 = \frac{1}{3}\left[K\left(\frac{1}{\gamma-\alpha} + \frac{2}{\beta-\gamma}\right)\right]^{-1}.
$$

If $\mathrm{Lip}(f) < \delta_0$ then

$$K\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)\mathrm{Lip}(f) < \frac{1}{3}, \tag{1.24}$$

and

$$\|G(f(y_1(\cdot))) - G(f(y_2(\cdot)))\|_\gamma \le \frac{1}{3}\|y_1-y_2\|_\gamma. \tag{1.25}$$

Thus, for any $\xi \in E_c$, by (1.20), we have

$$\|J(\xi, y_1) - J(\xi, y_2)\|_\gamma \le \frac{1}{3}\|y_1-y_2\|_\gamma, \tag{1.26}$$

as long as $\mathrm{Lip}(f) < \delta_0$.

By the Uniform Contraction Mapping Theorem, $J(\xi, \cdot\,)$ has a unique fixed point $y = x^*(t,\xi)$ for each $\xi \in E_c$. □

Lemma 1.8. *If* $\mathrm{Lip}(f) < \delta_0$, *then there exists a unique Lipschitz function* $\psi \in C_b^0(E_c, E_h)$ *such that*

$$W^c = \{x_c + \psi(x_c) \mid x_c \in E_c\}.$$

Proof. By Lemmas 1.7 and 1.6, (1.8) has a unique solution $x^*(t,\xi) = \tilde{x}(t, x^*(0,\xi))$, for any $\xi \in E_c$. By Lemma 1.6,

$$W^c = \{x^*(0,\xi) \mid \xi \in E_c\}.$$

Note that

$$x^*(0,\xi) = J(\xi, x^*(\cdot,\xi))(0) = \xi + \psi(\xi), \qquad \xi \in E_c,$$

where

$$\psi(\xi) = \int_\infty^t e^{-A\tau}\pi_u f(x^*(\tau,\xi))\,d\tau + \int_{-\infty}^0 e^{-A\tau}\pi_s f(x^*(\tau,\xi))\,d\tau. \tag{1.27}$$

We need to prove the boundedness and Lipschitz continuity of ψ.