Chapter 1

Bundles

In this chapter, we describe the basic structure upon which our study of jets will be based, namely that of bundles and sections. This structure is a generalisation of the more familiar structure of pairs of manifolds and maps, and allows more complicated topological arrangements. Although we shall be concerned primarily with local properties of jets, this more general description is still necessary for our discussion, because there are pairs of manifolds whose jet bundles do not themselves simplify to further pairs of manifolds.

1.1 Fibred Manifolds and Bundles

Many of the theories in modern mathematical physics can be described by considering smooth functions between differentiable manifolds. The domain of such a function might represent a region of space-time, and the codomain the possible states of the relevant physical system. Frequently, however, one considers not the function itself, but rather its *graph*: if the function is $f : M \longrightarrow F$ then its graph is the new function $\mathrm{gr}_f : M \longrightarrow M \times F$ defined by $\mathrm{gr}_f(p) = (p, f(p))$, and any function $\phi : M \longrightarrow M \times F$ which satisfies the condition $pr_1 \circ \phi = id_M$ is the graph of a uniquely-defined function f (namely, $f = pr_2 \circ \phi$). In this arrangement, the product manifold $M \times F$ is called the *total space*, because its local coordinate charts contain both dependent and independent variables for the function f. The domain M is also called the *base space*.

This way of looking at functions has two advantages. One is conceptual: the function may be thought of as a "field", in that for each point $p \in M$ there is a copy $\{p\} \times F$ of the codomain of f, and a single point in that copy gives the value of the field at p. This is a common way of picturing "vector fields", where the value of the field at a point may be represented by a vector attached to that point. The second advantage is more substantial, in that one may seek a generalisation of this arrangement where the total space as a whole is *not* diffeomorphic to the product of the base space and another manifold. For such a generalisation to be useful, however, there must nevertheless be a local product structure: each point of the total space must have a neighbourhood which "looks like" a product manifold. Such a structure is called a *fibred manifold*.

Definition 1.1.1 A *fibred manifold* is a triple (E, π, M) where E and M are manifolds and $\pi : E \longrightarrow M$ is a surjective submersion. E is called the *total space*, π the *projection*, and M the *base space*. For each point $p \in M$, the subset $\pi^{-1}(p)$ of E is called the *fibre over* p and is usually denoted E_p. ∎

As a shorthand, the same symbol E is sometimes used for the fibred manifold as for its total space. However this notation may be ambiguous, and in later chapters there will be many instances where the same manifold is the total space of two different fibred manifolds. We shall therefore denote the fibred manifold by the same symbol as we use for its projection, so that the shorthand for (E, π, M) will be π. Since the projection π of a fibred manifold (E, π, M) is a submersion, each connected component of the fibre E_p is a submanifold of E, and $\dim E_p = \dim E - \dim M$ is called the *fibre dimension* of π. We shall normally assume that both $\dim M$ and $\dim E_p$ are non-zero.

Example 1.1.2 If M and F are manifolds then $(M \times F, pr_1, M)$ is a fibred manifold. This is called a *trivial* fibred manifold; the word "trivial" has a technical meaning which is given in Definition 1.1.6. ∎

Example 1.1.3 Let $SL(2, \mathbf{R})$ be the three-dimensional manifold of real 2×2 matrices with determinant one, and let H be the subset $\operatorname{im} z > 0$ of the complex plane (regarded as a two-dimensional real manifold). Define a map $\pi : SL(2, \mathbf{R}) \longrightarrow H$ by

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{ai + b}{ci + d}.$$

A straightforward computation shows that the rank of π_* is 2 at each point of $SL(2, \mathbf{R})$. Since π is surjective, it follows that $(SL(2, \mathbf{R}), \pi, H)$ is a fibred manifold. ∎

Example 1.1.4 One of the simplest examples of a fibred manifold whose local product structure does not extend to a global product is the Möbius band. The total space (the Möbius band itself) may be constructed from the topological space $[0, 1] \times (0, 1)$ by identifying the points $(0, y)$ and

$(1, 1-y)$ and giving the quotient space the structure of a 2-dimensional smooth manifold in a straightforward way. The image of the set of points $[0,1] \times \{\frac{1}{2}\}$ under the quotient map is then diffeomorphic to the circle S^1, and the projection $[0,1] \times (0,1) \longrightarrow [0,1] \times \{\frac{1}{2}\}$ passes to the quotient to give the Möbius band the structure of a fibred manifold over the circle. Each fibre is just a copy of the open interval $(0,1)$, but the total space is not diffeomorphic to the Cartesian product $S^1 \times (0,1)$ because of the "twist". ∎

The justification for describing a fibred manifold as having a local product structure comes from the properties of submersions. By using the implicit function theorem, we may see that for each point $a \in E$ there is a neighbourhood $U_a \subset E$, some other manifold V_a, and a diffeomorphism

$$t_a : U_a \longrightarrow \pi(U_a) \times V_a$$

which satisfies the condition that $pr_1(t_a(b)) = \pi(b)$ for all $b \in U_a$. The condition on t_a asserts that the fibres of π (when restricted to U_a) correspond to the fibres of the Cartesian product projection pr_1. A condition such as this, involving the composition of maps, is often expressed by using a "commutative diagram":

$$\begin{array}{ccc} U_a & \xrightarrow{\quad t_a \quad} & \pi(U_a) \times V_a \\ \Big\downarrow {\scriptstyle \pi|_{U_a}} & & \Big\downarrow {\scriptstyle pr_1} \\ \pi(U_a) & \xrightarrow[\quad id \quad]{} & \pi(U_a) \end{array}$$

where $\pi|_{U_a}$ denotes the restriction of π to U_a. Such a diagram is meant to assert that, when there is more than one route between two different nodes, then all such routes give the same result. In this case, the assertion is simply that the two maps $pr_1 \circ t_a$ and $id \circ \pi|_{U_a}$ are equal.

The existence of a local product structure on the total space of a fibred manifold allows us to use special local coordinate systems called *adapted coordinates*. These correspond to the product coordinates which may be constructed on a product manifold $M \times F$ from coordinates on the individual manifolds M and F.

Definition 1.1.5 Let (E, π, M) be a fibred manifold such that $\dim M = m$, $\dim E = m+n$, and let $y : U \longrightarrow \mathbf{R}^{m+n}$ be a coordinate system on the open set $U \subset E$. The coordinate system y is called an *adapted coordinate system* if, whenever $a, b \in U$ and $\pi(a) = \pi(b) = p$, then $pr_1(y(a)) = pr_1(y(b))$ (where $pr_1 : \mathbf{R}^{m+n} \longrightarrow \mathbf{R}^m$). ∎

The meaning of this definition is that points in the same fibre $E_p \cap U$ have their first m coordinates equal, and are distinguished by their last n coordinates.

If $a \in E$ then adapted coordinates around a may be constructed from the local product structure in the following way. Starting with a coordinate system $x : W \longrightarrow \mathbf{R}^m$ around $\pi(a) = pr_1(t_a(a)) \in M$ (where W is chosen so that $W \subset \pi(U)$) and a coordinate system $u : V \longrightarrow \mathbf{R}^n$ around $pr_2(t_a(a)) \in V \subset V_a$, we define $y : t_a^{-1}(W \times V) \longrightarrow \mathbf{R}^{m+n}$ by

$$y = (x \circ pr_1 \circ t_a, u \circ pr_2 \circ t_a),$$

just as for product manifolds. Conversely, any adapted coordinate system $y : U \longrightarrow \mathbf{R}^{m+n}$ yields a coordinate system $x : \pi(U) \longrightarrow \mathbf{R}^m$ by setting $x(p) = pr_1(y(a))$, where $a \in E_p \cap U$; this is independent of the choice of a by Definition 1.1.5.

When dealing with the component functions of an adapted coordinate system, we shall usually adopt the following notation. If x^i $(1 \leq i \leq m)$ are the coordinate functions on M, then the coordinate functions on E will be labelled

$$(x^i, u^\alpha) \qquad 1 \leq i \leq m, \quad 1 \leq \alpha \leq n$$

so that the same symbol x^i will be used both for a function $\pi(U) \longrightarrow \mathbf{R}$ and for the composite function $U \longrightarrow \pi(U) \longrightarrow \mathbf{R}$. The latter function may also be written as the pullback $\pi^*(x^i)$, and this is the first of many occasions when the same symbol will be used to represent both an object and its pullback by a fibred manifold projection.

In many cases the idea of a fibred manifold without any additional restrictions, although useful, is slightly too general: for example, different fibres may have different topological structures. An example of this phenomenon may be constructed by taking the trivial bundle $(\mathbf{R} \times \mathbf{R}, pr_1, \mathbf{R})$

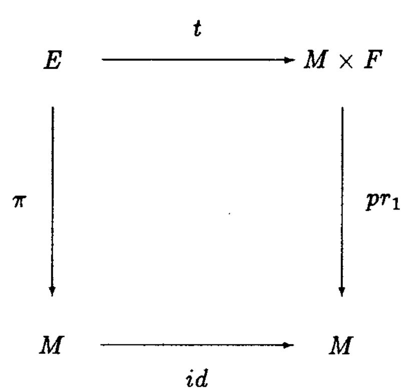
and deleting a single point. The result is a new fibred manifold where all the fibres except one are connected. If the fibred manifold is supposed to model a physical system then it may be unrealistic to allow the possible states of the system to depend on the choice of a particular point in space-time.

This problem may be resolved by insisting that the fibred manifold look rather more like a product than the definition of a submersion necessitates. The additional condition which such a fibred manifold must satisfy is expressed in terms of functions called *local trivialisations*, and the resulting object is called a *bundle*; after the present section, we shall be concerned almost entirely with bundles rather than more general fibred manifolds. We shall first describe what is meant by a *global* trivialisation.

Definition 1.1.6 If (E, π, M) is a fibred manifold then a (global) *trivialisation of* π is a pair (F, t) where F is a manifold (called a *typical fibre* of π) and $t : E \longrightarrow M \times F$ is a diffeomorphism satisfying the condition

$$pr_1 \circ t = \pi.$$

A fibred manifold which has at least one trivialisation is called *trivial*. ∎



In particular, our original example $(M \times F, pr_1, M)$ is a trivial fibred manifold using the identity map as the trivialisation. However, suppose $g : M \times F \longrightarrow F$ satisfies the condition that, for each $p \in M$, the map $g_p : F \longrightarrow F$ defined by $g_p(q) = g(p, q)$ is a diffeomorphism. Then the map $t : M \times F \longrightarrow M \times F$ defined by $t(p, q) = (p, g_p(q))$ is another trivialisation, so it is important to be clear that requiring a fibred manifold to be trivial does not give its total space the structure of a Cartesian product in any particular way. Nevertheless, the typical fibres corresponding to two different trivialisations must clearly be diffeomorphic, so referring to a typical fibre of π rather than of the trivialisation is justified.

Example 1.1.7 If the circle S^1 is regarded as the unit circle in $\mathbf{R}^2$, then we may define the map $\rho_1 : SL(2,\mathbf{R}) \longrightarrow S^1 \subset \mathbf{R}^2$ by

$$\rho_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{a}{\sqrt{a^2+c^2}}, \frac{c}{\sqrt{a^2+c^2}} \right),$$

and then

$$\begin{aligned} t_1 : SL(2,\mathbf{R}) &\longrightarrow H \times S^1 \\ t_1(A) &= (\pi(A), \rho_1(A)) \end{aligned}$$

is a diffeomorphism. Consequently t_1 is a trivialisation of the fibred manifold $(SL(2,\mathbf{R}), \pi, H)$. However, we may also define the map $\rho_2 : SL(2,\mathbf{R}) \longrightarrow S^1$ by

$$\rho_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{b}{\sqrt{b^2+d^2}}, \frac{d}{\sqrt{b^2+d^2}} \right),$$

and then

$$\begin{aligned} t_2 : SL(2,\mathbf{R}) &\longrightarrow H \times S^1 \\ t_2(A) &= (\pi(A), \rho_2(A)) \end{aligned}$$

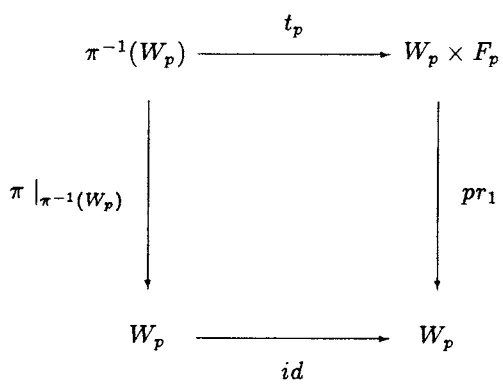
is another trivialisation of π. The existence of either trivialisation allows us to assert that π is trivial with typical fibre S^1. ∎

In the definition of a local trivialisation, the word "local" refers to the base manifold M rather than the total space E: the definition is concerned with expressing, in product form, subsets of E which are the unions of complete fibres of π.

Definition 1.1.8 If (E, π, M) is a fibred manifold and $p \in M$ then a *local trivialisation of π around p* is a triple (W_p, F_p, t_p) where W_p is a neighbourhood of p, F_p is a manifold and $t_p : \pi^{-1}(W_p) \longrightarrow W_p \times F_p$ is a diffeomorphism satisfying the condition

$$pr_1 \circ t_p = \pi|_{\pi^{-1}(W_p)} .$$

A fibred manifold which has at least one local trivialisation around each point of its base space is called *locally trivial* and is known as a *bundle*. ∎



It is worth noting that the existence of these local trivialisations around each point of M automatically implies that the map π is a submersion.

The concept of a typical fibre is also appropriate for bundles, although this is not quite immediate from the definition.

Lemma 1.1.9 *If (E, π, M) is a bundle then there is a manifold F such that, for each local trivialisation (W_p, F_p, t_p) of π, the manifolds F and F_p are diffeomorphic.*

Proof Notice first that if (W_p, F_p, t_p) and (W'_p, F'_p, t'_p) are both local trivialisations around the same point p then the manifolds F_p and F'_p must be diffeomorphic. So choose a fixed point $p \in M$ and a fixed local trivialisation (W_p, F_p, t_p), and put $F = F_p$. Let W be the set of points $q \in M$ such that there exists a local trivialisation (W_q, F, t_q) around q. Then W is non-empty, and is open because each W_q is open. On the other hand, $M - W$ must be open since it is the union of the open sets of points $r \in M$ where the local trivialisations (W_r, F_r, t_r) involve manifolds F_r which are *not* diffeomorphic to F. Therefore $M - W$ must be empty, because M is connected. ∎

On the total space of a bundle, adapted coordinate systems may be constructed from local trivialisations using coordinate systems on the base space and the typical fibre: this apparently unnecessary remark is useful when considering bundles with additional structure (such as vector bundles).

A trivial fibred manifold is obviously a bundle (and will be called a *trivial bundle*). The Möbius band is an example of a bundle which is not trivial. Further examples of bundles may be constructed from the manifolds of tangent and cotangent vectors associated with a given base manifold.

Example 1.1.10 Let TM denote the tangent manifold to the m-dimensional manifold M, and let $\tau_M : TM \longrightarrow M$ denote the map which associates to each tangent vector the point of M at which it is located. Then (TM, τ_M, M) is a bundle with typical fibre $\mathbf{R}^m$. To demonstrate this, it is convenient to use local coordinates. So let $\xi \in TM$ have the representation

$$\xi = \xi^i \left.\frac{\partial}{\partial x^i}\right|_p$$

where $p = \tau_M(\xi)$, the functions x^i are coordinate functions around p, and the summation convention is employed for the repeated index i. If $\gamma : \mathbf{R} \longrightarrow M$ is a curve whose tangent at zero is ξ then the real numbers ξ^i satisfy

$$\xi^i = (x^i \circ \gamma)'(0).$$

We may then define a coordinate system $(x^i, \dot{x}^i)$ on TM by writing (as usual) x^i instead of $x^i \circ \tau_M$, and setting $\dot{x}^i(\xi) = \xi^i$. To show that the fibred manifold constructed in this manner is locally trivial, let W_p be the coordinate neighbourhood of p on which the functions x^i are defined, and let $t : \tau_M^{-1}(W_p) \longrightarrow W_p \times \mathbf{R}^m$ be given by $t(\eta) = (\tau_M(\eta), \dot{x}(\eta))$. The map t is a diffeomorphism because it is the composition of the coordinate diffeomorphism $(x \circ \tau_M, \dot{x})$ on $\tau_M^{-1}(W_p)$ with the map $(x^{-1}, id_{\mathbf{R}^m})$. (The fact that TM has the topological properties which we require of a manifold, and that τ_M is therefore a bundle, is a consequence of a more general result which we shall give in Proposition 1.1.14.) ∎

Example 1.1.11 If $M = \mathbf{R}^m$ then $TM \cong \mathbf{R}^m \times \mathbf{R}^m$ and the tangent bundle τ_M is trivial. Indeed, if $x : M \longrightarrow \mathbf{R}^m$ is a global coordinate system on a manifold M then $(\tau_M, \dot{x}) : TM \longrightarrow M \times \mathbf{R}^m$ is a global trivialisation. ∎

Example 1.1.12 The tangent bundle (TS^1, τ_{S^1}, S^1) is trivial, even though the circle S^1 does not have a global coordinate system. To see this, let $\theta_1 : W_1 \longrightarrow \mathbf{R}$, $\theta_2 : W_2 \longrightarrow \mathbf{R}$ be two angle coordinate systems on S^1 whose domains W_1, W_2 together cover S^1, and such that if $p \in W_1 \cap W_2$ then $\theta_1(p) = \theta_2(p) \pm \pi$. Given a tangent vector $\xi \in TS^1$, suppose that ξ is determined by the curve γ, and put

$$\dot{\theta}(\xi) = (\theta_1 \circ \gamma)'(0)$$

if $\tau_{S^1}(\xi) \in W_1$,

$$\dot{\theta}(\xi) = (\theta_2 \circ \gamma)'(0)$$

if $\tau_{S^1}(\xi) \in W_2$. If it happens that $\tau_{S^1}(\xi) \in W_1 \cap W_2$ then $(\theta_1 \circ \gamma)'(0) = (\theta_2 \circ \gamma)'(0)$, because θ_1 and θ_2 differ by a constant, and so this procedure gives a well-defined mapping $\dot{\theta} : TS^1 \longrightarrow \mathbf{R}$. The map $(\tau_{S^1}, \dot{\theta}) : TS^1 \longrightarrow S^1 \times \mathbf{R}$ is then a global trivialisation. ∎

Example 1.1.13 The tangent bundle (TS^2, τ_{S^2}, S^2), where S^2 denotes the 2-sphere, is *not* trivial. To see this, suppose that there were a global trivialisation $t : TS^2 \longrightarrow S^2 \times \mathbf{R}^2$. Choose a non-zero element $v \in \mathbf{R}^2$, and define $X : S^2 \longrightarrow TS^2$ by

$$X(p) = t^{-1}(p, v).$$

Then $X(p)$ is a non-zero tangent vector in T_pS^2 which depends smoothly on p, and so X is a non-vanishing smooth vector field on S^2: but this contradicts the famous Hairy Ball Theorem. ∎

An important property of any bundle is that the manifold structure on its total space E is completely determined by the manifold structures on its base space M and typical fibre F. For a trivial bundle $(M \times F, pr_1, M)$ this is a familiar result, but it applies equally to the case where the bundle is not trivial. The reason for this is that, if (W, F, t) is a local trivialisation, then t transports the manifold structure from $W \times F$ to the "strip" $\pi^{-1}(W)$ of E, and where the strips overlap the manifold structures are the same. In fact this technique can be used to construct a manifold structure on E when it is not given *a priori*.

Proposition 1.1.14 *Let M and F be manifolds, E a set, and $\pi : E \longrightarrow M$ a function such that, for each $p \in M$, $\pi^{-1}(p)$ has the structure of an n-dimensional manifold. Suppose also that, for each $p \in M$, there is a neighbourhood W_p of p and a bijection $t_p : \pi^{-1}(W_p) \longrightarrow W_p \times F$ satisfying:*

1. $pr_1 \circ t_p = \pi|_{\pi^{-1}(W_p)}$;
2. *for each $q \in W_p$, $pr_2 \circ t_p|_{\pi^{-1}(q)} : \pi^{-1}(q) \longrightarrow F$ is a diffeomorphism.*

Then E may be given a unique structure as a manifold such that π becomes a bundle and the maps t_p become local trivialisations.

Proof Let $a \in \pi^{-1}(p)$ and let $x : W \longrightarrow \mathbf{R}^m$ be a coordinate system around p and $u : V \longrightarrow \mathbf{R}^n$ be a coordinate system around $pr_2(t_p(a)) \in F$. Then, with our usual understanding about domains of functions being sufficiently small, the map $y_p = (x, u) \circ t_p$ is a "coordinate system" around a. We shall show that, whenever the domains of y_p and y_q have non-empty intersection then $y_q \circ y_p^{-1}$ is smooth (and hence a diffeomorphism). Since each map (x, u) is a diffeomorphism, it will be sufficient to show that

$$t_q \circ t_p^{-1} : (W_p \cap W_q) \times F \longrightarrow (W_p \cap W_q) \times F$$

is smooth. To do this, we note first that for each $r \in W_p \cap W_q$, the map $t_q \circ t_p^{-1}\big|_{\{r\} \times F}$ induces a diffeomorphism of F with itself. A consequence of this is that the map

$$\begin{array}{rcl}(W_p \cap W_q) \times F & \longrightarrow & F \\ (r, c) & \longmapsto & pr_2\left(t_q \circ t_p^{-1}\big|_{\{r\} \times F}(c)\right)\end{array}$$

is also smooth. But this latter map is just the second component of $t_q \circ t_p^{-1}$, and the first component of $t_q \circ t_p^{-1}$ is simply $pr_1 : (W_p \cap W_q) \times F \longrightarrow W_p \cap W_q$. Therefore $t_q \circ t_p^{-1}$ is smooth, and so E acquires a finite-dimensional C^∞ atlas. The uniqueness of this manifold structure follows because, if each function t_p is a diffeomorphism for two manifold structures on E, then id_E is a diffeomorphism between the two manifold structures. It now follows immediately that the map π is smooth because locally it is just $pr_1 \circ t_p$, and it is obviously surjective. The functions t_p therefore become local trivialisations for the bundle (E, π, M).

We may also show that E satisfies the topological conditions which we require of a manifold. First we shall demonstrate the Hausdorff property. So let $a, b \in E$. If $\pi(a) \neq \pi(b)$ then there are open sets $W_a, W_b \subset M$ which separate $\pi(a)$ and $\pi(b)$, so that $\pi^{-1}(W_a)$, $\pi^{-1}(W_b)$ will separate a and b. On the other hand, if $\pi(a) = \pi(b)$ ($= p$, say) then $pr_1(t_p(a)) = pr_1(t_p(b))$ so that $pr_2(t_p(a)) \neq pr_2(t_p(b))$ since t_p is bijective. Then there must be open sets $V_a, V_b \subset F$ which separate $pr_2(t_p(a))$ and $pr_2(t_p(b))$, and therefore open sets $(pr_2 \circ t_p)^{-1}(V_a)$, $(pr_2 \circ t_p)^{-1}(V_b)$ which separate a and b.

Next we shall show that E is second-countable. To do this, we shall first demonstrate that there is a countable family of local trivialisations whose neighbourhoods W_p cover M. So let X_λ be a countable basis for the open sets in M. For each $q \in M$, choose an open set X_{λ_q} such that $q \in X_{\lambda_q} \subset W_q$ and consider the triple

$$\left(X_{\lambda_q}, F, t_q|_{\pi^{-1}(X_{\lambda_q})}\right).$$

Since there are only countably many different open sets X_{λ_q}, we may choose, for each such set, one particular $p \in M$ which gives rise to it and hence obtain the required countable family

$$\left(X_p, F, t_p|_{\pi^{-1}(X_p)}\right).$$

Consequently any open set $O \subset E$ may be written as a countable union

$$O = \bigcup_p (O \cap \pi^{-1}(X_p))$$

where $O \cap \pi^{-1}(X_p)$ is diffeomorphic to an open subset of $X_p \times F$. Since each product manifold $X_p \times F$ has a countable basis of open sets, it follows that E does as well.