SOME RECENT RESULTS FOR THE PLANAR ISING MODEL

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Introduction

The planar Ising model has become one of the most important statistical mechanical systems for the study of phase transitions and critical phenomena. Although there are many rigorous results, such as correlation inequalities, Peierls argument and the Yang-Lee circle theorem to name but three [reviewed by Griffiths, 1971], which are dimension-independent in their validity and which lead to results of considerable interest, only the planar model to date benefits from the added insights which stem ultimately from Onsager´s tour de force [Onsager, 1944]. It is not the purpose of this article to enter into a general review - for this the reader is referred elsewhere [Gallavotti, 1972] but rather to discuss two more mathematical aspects of the development of Onsager´s solution. The first item is the Yang-Baxter system of equations for the planar Ising model in zero field with transfer in the (1,1) direction. This work shows that the Clifford-algebraic structure of the exact solution is a natural consequence of the star-triangle equations. The second item is a Fredholm system which turns out to be of crucial importance in understanding surface and interface problems, as well as the pair correlation function.

Consider a planar lattice drawn on $\mathbb{Z}^2$ and denoted by $\Lambda(N,M) = \{(x,y): -N \le x \le N, -M \le x \le M\}$. At each lattice site (x,y) there is a spin variable $\sigma(x,y) = \pm 1$. The specification of all such spin variables on $\Lambda(N,M)$ is called a spin configuration denoted $\{\sigma\}_{N,M}$. It has an energy

$$E_\Lambda(\{\sigma\}_{N,M}) = - \sum_{|\underline{i}-\underline{j}|=1} J(|\underline{i}-\underline{j}|)\,\sigma(\underline{i})\,\sigma(\underline{j}) - H\,\Sigma\,\sigma(\underline{i}) \tag{1.1}$$

The summations are over points of $\Lambda(N,M)$. We have $J(|\underline{k}|) > 0$ for $|\underline{k}| = 1$; this is the ferromagnetic case. It will turn out to be useful to introduce boundary fields $H(\underline{i})$ which supplement the constant field H. They produce a boundary energy

$$E_B(\{\sigma\}_{N,M}) = - \sum_{\underline{i}\in\partial\Lambda} H(\underline{i})\,\sigma(\underline{i}) \tag{1.2}$$

The canonical configurational probability is

$$p_\Lambda(\{\sigma\}_{N,M}) = Z_\Lambda^{-1} \exp - \beta\, E_\Lambda(\{\sigma\}_{N,M}) \tag{1.3}$$

where β is the inverse temperature in units of the Boltzmann constant and Z_Λ normalises (1.3).

Consider the one-particle correlation function $\langle\sigma(0,0)\rangle(B,N,M)$ where $\langle . \rangle$ is expectation with respect to the measure (1.3) and B denotes that boundary fields have been included. First, let us consider the case $H = 0$. At low enough temperatures, the Peierls argument, which is reviewed by Griffiths [1971] shows that

$$\lim_{N,M\to\infty} \langle\sigma(0,0)\rangle\,(+,N,M) > 0 \tag{1.4}$$

where + means that all spins on the boundary $\partial\Lambda$ are fixed to be +1 . Since

$$\lim_{N,M\to\infty} \langle\sigma(0,0)\rangle (+,N,M) = - \lim_{N,M\to\infty} \langle\sigma(0,0)(-,N,M) \quad (1.5)$$

by an obvious symmetry (recall that $H = 0$), the system is unstable with respect to its boundary specification; this is one aspect of phase transition. Onsager´s approach may be developed to give the left hand side of (1.4), which is called the spontaneous magnetisation, for all values of β, in particular those near the critical value βc [Abraham and Martin-Löf 1973].

Another example is as follows: no matter how the boundary conditions B are chosen, we cannot have translationally-variant correlation functions in the planar Ising model in the limit where the boundaries recede to infinity (with mild technical restrictions). This result is due to Aizenman [1979,1980] and to Higuchi [1979] and is true for all β . It can be shown that if the spins are +1 round the boundary above the line $i_2 = 0$ but -1 below, then the magnetisation satisfies the limiting law [Abraham and Reed, 1976]

$$\lim_{N\to\infty} \lim_{M\to\infty} \langle\sigma(\delta N,\alpha M^{1/2})\rangle = m^{*}\mathrm{sgn}\alpha \quad \Phi\left(\frac{b|\alpha|}{\sqrt{1-\delta^2}}\right) \quad (1.6)$$

for all $|\delta| < 1$, where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \quad (1.7)$$

and

$$b = (\sinh 2(K-K^{*}))^{1/2} \quad (1.8)$$

This is less general, but contains more information pertinent to physics, in particular on how the traditional van der Waals-Maxwell picture breaks down. In this picture, the interface between the two phases which coexist (for H=0) for $\beta > \beta_c$ is localised in laboratory-fixed axes.

The n-point correlation functions for local observables $0_1 \dots 0_n$ which just act on single rows of spins are written as

$$<0_1(\underline{r}_1) \dots 0_n(\underline{r}_n)> (B,N.M)$$

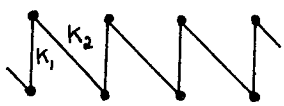
$$= \frac{<\phi_1|V^{x_1} 0_1(y_1) V^{x_2-x_1} 0_2(y_2) \dots V^{(x_n-x_{n-1})} 0_n(y_n) V^{N-x_n}|\phi_2>}{<\phi_1|V^N|\phi_2>} \quad (1.9)$$

In the above, V is the transfer matrix [Schultz et al 1964; Abraham 1978a)] which has been symmetrised for spectral convenience. Also, the independence of V on x means that the boundary conditions B at the points $y = \pm m$ are translationally invariant. In what follows, we shall take cyclic boundary conditions although this is not necessary [Abraham and Martin-Löf 1973], but the symmetry is very convenient, particularly in sections 3 and 4 below. The vectors $|\phi_j>$ describe the faces of the cylinder; the scalar products produce the desired weighted sums over end-states provided the $|\phi_j>$ are chosen correctly [see, for instance Abraham and Reed, 1976 for the derivation of (1.6)]. Since N and M are finite, (1.9) makes perfect sense. It can be analysed by knowing the spectrum of V and matrix elements of the local operators $0_j(y)$ in the basis of eigenvector of V . The first follows from Onsager [1944] and simplifications due to Kaufman [1949], Onsager and Kaufman

[1949] and finally Schultz, Mattis and Lieb [1964]. Since then, the matrix elements have been found [Abraham, 1978 a,b,c]. The only remaining problem is the evaluation of scalar products from the eigenvector basis to the $|\phi_j\rangle$. Sometimes the $|\phi_j\rangle$ are rather atypical of the equilibrium state, so that great care is needed with the $M \to \infty$ limit.

2. Monodromy and Yang-Baxter equations

Let us consider the transfer matrix T which works in the (1.1) direction:



The couplings in units of β are denoted K_j and are arranged as shown.

Each diagonal bond between rows is decorated with an auxiliary spin labelled τ using the equation

$$e^{K_2\sigma_1\sigma_2} = \rho \sum_{\tau} e^{(L_1\sigma_1 + L_2\sigma_2)\tau} \tag{2.1}$$

Thus the transfer matrix is

$$T = \mathrm{Tr}\, \mathbb{M} \tag{2.2}$$

where the trace is taken over auxiliary spins and the monodromy matrix is

$$\mathbb{M} = \prod_{1}^{n} L_j \tag{2.3}$$

where the product is a matrix one over auxiliary variables which is ordered with increasing j to the right and

$$L_j(\tau_j, \tau_{j+1}) = \exp(K_1\sigma_j\sigma_j' + L_1\sigma_j\tau_j + L_2\sigma_j'\tau_{j+1}) \tag{2.4}$$

Thus we may think of the L_j as 2 x 2 matrices whose entries themselves are matrix-valued in the variables which describe the original rows of spins; these spins are normally referred to as quantum ones. Each element of L_j is now taken as a matrix representative of an operator in the two-dimensional complex vector space $\mathbb{C}^2$, we deal with the product in (2.3) in a tensor product space

$$\mathfrak{h}_M = \bigotimes_{1}^{M} \mathbb{C}^2 \tag{2.5}$$

using the Pauli spin operators

$$\sigma_j^{\alpha} = \bigotimes_{1}^{j-1} 1 \otimes \sigma^{\alpha} \bigotimes_{j+1}^{M} 1 \tag{2.6}$$

where σ^{α} are the usual Pauli matrices.

After some algebra [Abraham and Davies, 1988], (2.3) and (2.4) become

$$\mathbb{M} = (2 \sinh 2K_1)^{M/2} \begin{pmatrix} \cosh K_2 \\ i \sinh K_2 \sigma_1^x \end{pmatrix} \tilde{T}(1, - \sigma_M^x) \tag{2.7}$$

where

$$\tilde{T} = \prod_{1}^{2M-1} U_j \tag{2.8}$$

with

$$U_{2j-1} = \exp - K_1^* \sigma_j^z \tag{2.9}$$

$$U_{2j} = \exp K_2 \sigma_j^x \sigma_{j+1}^x \tag{2.10}$$

and the matrices L_j are recovered in the representation with σ_j^x diagonal. The operator (2.7) can be analysed using the Jordan-Wigner transformation

$$f_j^{\dagger} = P_{j-1}(\sigma_j^x + i\sigma_j^y)/2 \tag{2.11}$$

with

$$P_0 = 1, \quad P_k = \prod_{1}^{k} (-\sigma_j^z) \tag{2.12}$$

The associated spinors

$$\Gamma_{2j-1} = f_j^{\dagger} + f_j, \quad \Gamma_{2j} = - i (f_j^{\dagger} - f_j) \tag{2.13}$$

generate a Clifford algebra through the anticommutation relations

$$[\Gamma_j, \Gamma_k]_+ = 2\delta_{jk} \tag{2.14}$$

which follow because

$$[f_j, f_k]_+ = 0, \quad [f_j, f_k^\dagger]_+ = \delta_{jk} \tag{2.15}$$

as is easily checked.

The factors U_j become spinor rotations

$$U_{2j-1} = \exp i K_1^* \Gamma_{2j-1}\Gamma_{2j} \tag{2.16}$$

$$U_{2j} = \exp i K_2 \Gamma_{2j} \Gamma_{2j+1} \tag{2.17}$$

and

$$\mathbb{M} = (2 \sinh 2K_1)^{M/2} \begin{pmatrix} \cosh K_2 \\ - \sinh K_2 \Gamma_1 \end{pmatrix} \tilde{T}(1, - i\Gamma_{2M}P_M) \tag{2.18}$$

Thus we have

$$T = (2 \sinh 2K_1)^{M/2} \{\cosh K_2 \tilde{T} + i \sinh K_2 \Gamma_1 \tilde{T} \Gamma_{2M}P_M\} \tag{2.19}$$

Returning to (2.12), it is clear that

$$[T, P_M]_- = 0 \tag{2.20}$$

which can be thought of as a statement of rotational invariance through an angle π about the Z-axis, or equally well, of conservation of parity of fermion number. Introducing the projectors

$$Q_\pm = (1 \pm P_M)/2 \tag{2.21}$$

onto the sub-spaces $\mathfrak{h}_M^{\pm}$ with even/odd numbers of fermions gives

$$T = T_+Q_+ + T_-Q_- \tag{2.22}$$

with

$$T_\pm = (2 \sinh 2K_1)^{M/2} \{\cosh K_2 \tilde{T} + i \sinh K_2 \Gamma_1 \tilde{T} \Gamma_{2M} \} \tag{2.23}$$

which can be brought to diagonal form using the Schultz-Mattis-Lieb [1964] or Kaufman-Onsager [1949] procedures. The only difference from the usual case is that

(2.23) is a sum of spinor rotations, up to a factor. After some algebra [Abraham and Davies 1988], the result is

$$T_{\pm} = \prod_{\omega\in\Omega_{M}(\mp)} (c_1c_2 + |1 - s_1s_2e^{i\omega}|)^{1/2} \exp - \sum_{\omega\in\Omega_{M}(\mp)} \gamma(\omega)G^{+}(\omega)G(\omega) \qquad (2.24)$$

where

$$\Omega_M(\pm) = \{\omega \in (-\pi,\pi],\ \exp iM\omega = \pm 1\} \qquad (2.25)$$

and

$$G^{+}(\omega) = U_{\pm}F(\omega)^{+}U_{\pm}^{+}, \quad \omega \in \Omega_M(\pm) \qquad (2.26)$$

with

$$F^{+}(\omega) = M^{-1/2}\sum_{1}^{M} e^{i\omega j} f_j^{+} \qquad (2.27)$$

and

$$U_{\pm} = \begin{Bmatrix} 1 \\ U_0 \end{Bmatrix} \exp i \sum_{\omega\in(0,\pi)\cap\Omega_M(\mp)} \Theta(\omega)(F^{+}(-\omega)F^{+}(\omega) + F(\omega)F(-\omega)) \qquad (2.28)$$

with

$$U_0 = \exp i\,\Theta(0)(F(0)^{+} + F(0)) \qquad (2.29)$$

The transformation angle is given by

$$e^{2i\Theta(\omega)} = (1 - s_1s_2e^{i\omega})/|1 - s_1s_2e^{i\omega}| \qquad (2.30)$$

and the eigenvalues are constructed in terms of

$$e^{-\gamma(\omega)} = (c_1c_2 - |1 - s_1s_2e^{i\omega}|)/(s_1 + s_2e^{i\omega}) \qquad (2.31)$$

(note $|\exp - (\omega)| < 1$ by choice).

In (2.30) and (2.31) we use the convenient notation

$$c_j = \cosh 2K_j \quad s_j = \sinh 2K_j \quad j = 1,2 \qquad (2.32)$$

It is interesting to note directly from (2.30) that transfer matrices with the same value of $k = 1/s_1s_2$ form a commuting family; this was known to Onsager [1944, 1971] and to Stephen and Mittag [1972]. The thermodynamic critical point is given by $k = 1$; the nature of the spectrum clearly depends crucially on the sign of $(k - 1)$: if $k > 1$, $\exp 2i\theta(0) = 1$ so $U_0 = 1$ whereas if $k < 1$ (low-temperature) $\exp 2i\theta(0) = -1$ so that $U_0 = i(F_0^+ + F_0)$. Reference to (2.21) to (2.23) allows one to construct the spectrum. Notice that for $k < 1$ there are no states with an odd number of $G^+(\omega)$-created fermions.

The behaviour of the transformation angle mentioned above gives the function

$$\Theta(e^{i\omega}) = \exp(-2i\theta(\omega)) \tag{2.33}$$

an interesting behaviour; the winding number

$$I(\Theta) = \frac{1}{2\pi} \Delta_{|z|=1}(\arg \Theta(z)) \tag{2.34}$$

has the value zero for $k > 1$ but -1 if $k < 1$. This turns out to have important consequences for the matrix-element theory described in the next section.

Let us return meanwhile to (2.7); the reader familiar with the quantum inverse scattering method (QISM : the reader is referred for reviews to Takhtadzhan and Fadeev [1979] and to Thacker [1981] will find the matrix structure in the auxiliary variables suggestive. In the QISM, the off-diagonal elements of $\mathbb{M}$ generate the eigenvector of T by behaving somewhat like ladder operators acting on a vacuum. It turns out that if k is held fixed, but the ratio $-s_2/s_1$ is given a unimodular complex value $e^{i\omega}$, then

$$G^+(\omega) = \alpha\, \mathbb{M}_{21} + \beta\, \mathbb{M}_{12} P_M \tag{2.35}$$

where α and β are given by tedious algebra. It is easy to confirm that the G-operators have c-number-valued anticommutators which only become fermionic for a suitable choice of the ω. Thus we expect M_{12} and M_{21} to have an associated algebra; in the QISM it is natural to inquire whether such a structure can be obtained directly from Yang-Baxter equations [Baxter, 1982]. These are of the form

$$L_j \otimes L_j' = L_j' \otimes L_j \qquad (2.36)$$

with R non-singular. Here the direct product is in auxiliary space, the order of quantum operators is respected and R is a 4 x 4 matrix which must be found. Its existence is assured [Gutkin, 1986] since we have a commuting family of transfer matrices; all we need to construct it is the star-triangle relation discovered by Onsager [Onsager 1944, Baxter 1982]. This work is discussed in much more detail elsewhere [Abraham and Davies, 1988]. The point we wish to make here is that (2.36) does not establish a serious alternative calculational procedure to the ones used, rather it explains in a simple way why the Jordan-Wigner transformation is encountered since the algebra of the $\mathbb{M}_{ij}$ can be got.

Ultimately the techniques of the next section might be extended to get the n-point functions of the vertex models of Lieb [for a review, see Lieb and Wu 1971] and of Baxter [Baxter 1982, and references therein].

3. Matrix elements

Let us now turn to the problem of the matrix elements $\langle \phi_j | 0 | \phi_k \rangle$ which will be required in the spectral decomposition of (1.9) if $[0, P_M]_- = 0$ then matrix elements are only non-zero if $\phi_j, \phi_k \in \mathfrak{h}_M^+$ or $\phi_j, \phi_k \in \mathfrak{h}_M^-$. Such an 0 will be a polynomial of even degree in the Fermi operators and the matrix elements can be obtained in principle at least, from the Wick theorem. A typical example here is the