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## K-Theory For Discrete Groups

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Let  $X$  be a  $C^\infty$  manifold.

Suppose given

(1) A  $C^\infty$  foliation  $F$  of  $X$

or (2) A  $C^\infty$  (right) action of a Lie group  $G$  on  $X$ ,  
 $X \times G \rightarrow X$

or (3) A  $C^\infty$  (right) action of a discrete group  
 $\Gamma$  on  $X$ ,  $X \times \Gamma \rightarrow X$ .

In (2)  $G$  is a Lie group with  $\pi_0(G)$  finite.

In (3) "discrete" simply means that  $\Gamma$  is a group topologized by the discrete topology in which each point is an open set.

Each of these three cases gives rise to a  $C^*$  algebra  $A$

(1)  $A = C^*(X, F)$ , the foliation  $C^*$  algebra [10] [12]

(2)  $A = C_0(X) \rtimes G$ , the reduced crossed-product  $C^*$ -algebra resulting from the action of  $G$  on  $C_0(X)$ . As usual  $C_0(X)$  denotes the abelian  $C^*$  algebra of all continuous complex-valued functions on  $X$  which vanish at infinity.

(3)  $A = C_0(X) \rtimes \Gamma$ , the reduced crossed-product  $C^*$ -algebra resulting from the action of  $\Gamma$  on  $C_0(X)$ .

In [4] [6] we defined for each case (and also for crossed-products twisted

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by a 2-cocycle) a geometric K theory and a map  $\mu$  from the geometric K theory to the K theory of the relevant  $C^*$ -algebra:

- (1)  $\mu : K^i(X, F) \rightarrow K_i[C^*(X, F)]$
- (2)  $\mu : K^i(X, F) \rightarrow K_i[C_0(X) \rtimes G]$
- (3)  $\mu : K^i(X, \Gamma) \rightarrow K_i[C_0(X) \rtimes \Gamma] \quad i = 0, 1 \dots$

The map  $\mu$  in essence assigns to a symbol the index of its underlying elliptic operator. In [4] [6] we conjectured that  $\mu$  is always an isomorphism. At the present time this conjecture is still alive.

Over the long run it will, of course, be fascinating to see whether such an extremely general conjecture can endure.

In this note we shall briefly review the state of the art for the special case of a discrete group  $\Gamma$  operating on a point. The  $C^*$ -algebra is then  $C_r^*\Gamma$ , the reduced  $C^*$ -algebra of  $\Gamma$ .

On the positive side there is the beautiful Mayer-Vietoris exact sequence of M. Pimsner [27]. This exact sequence includes earlier results of Pimsner-Voiculescu [28], T. Natsume [26], C. Lance [23], J. Cuntz [14], and J. Anderson-W. Paschke [2]. Using his bivariant KK theory, G.G. Kasparov [19] [20] [21], has verified the conjecture for discrete subgroups of  $SO_0(n, 1)$ . Also on the positive side is the transverse fundamental class result of [13]. We shall report further on this in [7].

On the negative side there are the recent examples of G. Skandalis [39]. These examples at least indicate that life is not as simple as one might have hoped. Also on the negative side is the utter lack of a Kunneth theorem and of any calculated examples where the discrete group  $\Gamma$  is infinite and has Kazhdan's property  $T$ . In addition there is the somewhat disturbing observation of M. Gromov that mathematicians have never proved a non-trivial theorem about all discrete groups.

It is a pleasure to thank J. Anderson, D. Burghlea, M. Gromov, J. Mingo, and G. Skandalis for enlightening and informative conversations. Also, we thank our British hosts for very lively and stimulating meetings at the Universities of Durham and Warwick during the spring and summer of 1987.

**§1. Statement of the Conjecture**

For simplicity assume that  $\Gamma$  is countable. (For  $\Gamma$  uncountable see Appendix 2 of [6].)  $\mathfrak{C}(\cdot, \Gamma)$  denotes the category of all proper  $C^\infty$   $\Gamma$ -manifolds. Thus an object of  $\mathfrak{C}(\cdot, \Gamma)$  is a  $C^\infty$  manifold  $W$  with a given proper (right)  $C^\infty$  action of  $\Gamma$  :

$$(1.1) \quad W \times \Gamma \rightarrow W$$

$W$  is an ordinary  $C^\infty$  manifold.  $W$  is Hausdorff, finite dimensional, second countable, and without boundary. Each  $\gamma \in \Gamma$  acts on  $W$  by a diffeomorphism. Recall that the action (1.1) is proper if and only if the map

$$(1.2) \quad W \times \Gamma \rightarrow W \times W$$

which takes  $(w, \gamma)$  to  $(w, w\gamma)$  is proper in the usual sense, i.e. the inverse image of any compact set in  $W \times W$  is a compact set in  $W \times \Gamma$ . This implies that each isotropy group of the action (1.1) is finite, and that the quotient space  $W/\Gamma$  is a Hausdorff second countable orbifold.

A morphism in  $\mathfrak{C}(\cdot, \Gamma)$  is a  $C^\infty$   $\Gamma$ -equivariant map  $f : W_1 \rightarrow W_2$ .  $f$  is not required to be proper. Associated to  $f$  is a homomorphism of abelian groups

$$(1.3) \quad f_i : K_i[C_0(TW_1) \rtimes \Gamma] \rightarrow K_i[C_0(TW_2) \rtimes \Gamma]$$

$i = 0, 1$

In (1.3)  $TW_j$  is the tangent bundle of  $W_j$ .  $C_0(TW_j) \rtimes \Gamma$  is the crossed-product  $C^*$ -algebra resulting from the evident action of  $\Gamma$  on  $TW_j$ . To define  $f_i$  first note that  $TW_j$  is an even-dimensional almost-complex manifold. The derivative map

$$(1.4) \quad f' : TW_1 \rightarrow TW_2$$

is, therefore, a K-oriented map and so yields

$$(f') \in KK^0(C_0(TW_1), C_0(TW_2)).$$

Consider the diagram

$$\begin{array}{ccc} KK^0_\Gamma(C_0(TW_1), C_0(TW_2)) & \rightarrow & KK^0(C_0(TW_1), C_0(TW_2)) \\ \downarrow & & \\ KK^0(C_0(TW_1) \rtimes \Gamma, C_0(TW_2) \rtimes \Gamma) & & \\ \downarrow & & \\ \text{Hom}_{\mathbb{Z}}(K_*[C_0(TW_1) \rtimes \Gamma], K_*[C_0(TW_2) \rtimes \Gamma]) & & \end{array}$$

in which the horizontal arrow is the forgetful map and the two vertical arrows are as in Kasparov [19] [20]. Since all structures involved in defining

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$(f') \in KK^0(C_0(TW_1), C_0(TW_2))$  are  $\Gamma$ -equivariant  $(f')$  lifts to give  $[f'] \in KK^0_\Gamma(C_0(TW_1), C_0(TW_2))$ . The horizontal arrow in (1.5) sends  $[f']$  to  $(f')$ . The map  $f_1$  of (1.3) is then obtained by applying the two vertical arrows to  $[f']$ .

(1.6) **Lemma:**  $f_1 : K_i[C_0(TW_1) \rtimes \Gamma] \rightarrow K_i[C_0(TW_2) \rtimes \Gamma]$  depends only on the homotopy class (as a  $C^\infty$   $\Gamma$ -equivariant map) of  $f$ .

(1.7) **Lemma:** Let  $f : W_1 \rightarrow W_2$  and  $g : W_2 \rightarrow W_3$  be morphisms in  $\mathcal{C}(\cdot, \Gamma)$ . Then  $(gf)_1 = g_1 f_1$ .

(1.8) **Definition:** 
$$K^i(\cdot, \Gamma) = \lim_{\mathcal{C}(\cdot, \Gamma)} K_i[C_0(TW) \rtimes \Gamma] \quad i = 0, 1$$

**Remarks:** In (1.8) the limit is taken using the  $f_1$  maps of (1.3). Let  $F_i(\cdot, \Gamma)$  be the free abelian groups generated by all<sup>1</sup> pairs  $(W, \xi)$  where  $W$  is an object of  $\mathcal{C}(\cdot, \Gamma)$  and  $\xi \in K_i[C_0(TW) \rtimes \Gamma]$ .  $R_i(\cdot, \Gamma)$  denotes the subgroups of  $F_i(\cdot, \Gamma)$  generated by all elements of the form

- (i)  $(W, \xi + \eta) - (W, \xi) - (W, \eta)$
- (ii)  $(W_1, \xi) - (W_2, f_1 \xi)$

Then definition (1.8) is:

(1.9) 
$$K^i(\cdot, \Gamma) = F_i(\cdot, \Gamma) / R_i(\cdot, \Gamma) \quad i = 0, 1$$

For each object  $W$  of  $\Gamma$  we have the homomorphism of abelian groups

(1.10) 
$$\psi_W : K_i[C_0(TW) \rtimes \Gamma] \rightarrow K^i(\cdot, \Gamma)$$

defined by

(1.11) 
$$\psi_W(\xi) = (W, \xi) \quad \xi \in K_i[C_0(TW) \rtimes \Gamma]$$

For any morphism  $f : W_1 \rightarrow W_2$  in  $\mathcal{C}(\cdot, \Gamma)$ , the diagram

<sup>1</sup> To avoid set-theoretic difficulties assume that  $W$  is a  $C^\infty$  sub-manifold (which is a closed subset) of some Euclidean space  $\mathbb{R}^m$ . This is possible by the Whitney embedding theorem.

$$(1.12) \quad \begin{array}{ccc} & f_1 & \\ & \downarrow & \\ K_i[C_0(TW_1) \rtimes \Gamma] & \rightarrow & K_i[C_0(TW_2) \rtimes \Gamma] \\ & \searrow \varphi_{W_1} & \swarrow \varphi_{W_2} \\ & K^i(\cdot, \Gamma) & \end{array}$$

commutes.

$K^i(\cdot, \Gamma)$  has the following universal property. Let  $H$  be an abelian group. Suppose given for each object  $W$  of  $\mathfrak{C}(\cdot, \Gamma)$  a homomorphism of abelian groups

$$(1.13) \quad \psi_W : K_i[C_0(TW) \rtimes \Gamma] \rightarrow H$$

Assume that for each morphism  $f : W_1 \rightarrow W_2$  in  $\mathfrak{C}(\cdot, \Gamma)$  the diagram

$$(1.14) \quad \begin{array}{ccc} & f_1 & \\ & \downarrow & \\ K_i[C_0(TW_1) \rtimes \Gamma] & \rightarrow & K_i[C_0(TW_2) \rtimes \Gamma] \\ & \searrow \psi_{W_1} & \swarrow \psi_{W_2} \\ & H & \end{array}$$

commutes.

Then there is a unique homomorphism of abelian groups

$$(1.15) \quad \psi : K^i(\cdot, \Gamma) \rightarrow H$$

with  $\psi_W = \psi \circ \varphi_W$  for each object  $W$  of  $\mathfrak{C}(\cdot, \Gamma)$ .

(1.15) applies to determine a homomorphism

$$(1.16) \quad \mu : K^i(\cdot, \Gamma) \rightarrow K_i[C_r^*\Gamma].$$

For each object  $W$  of  $\mathfrak{C}(\cdot, \Gamma)$  the Dirac operator of  $TW$  is an element of

$KK_\Gamma^0(C_0(TW), \mathbb{C})$ . Applying Kasparov's map

$$(1.17) \quad KK_\Gamma^0(C_0(TW), \mathbb{C}) \rightarrow KK^0(C_0(TW) \rtimes \Gamma, C_r^0\Gamma)$$

we then obtain a homomorphism of abelian groups

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$$(1.18) \quad \mu_W : K^i[C_0(TW) \rtimes \Gamma] \rightarrow K_i[C_r^*\Gamma]$$

(1.19) **Lemma:** For each morphism  $f : W_1 \rightarrow W_2$  in  $\mathcal{C}(\cdot, \Gamma)$  there is commutativity in the diagram

$$\begin{array}{ccc}
 & f_1 & \\
 K_i[C_0(TW_1) \rtimes \Gamma] & \xrightarrow{\quad} & K_i[C_0(TW_2) \rtimes \Gamma] \\
 \mu_{W_1} \swarrow & & \searrow \mu_{W_2} \\
 & K_i[C_r^*\Gamma] &
 \end{array}$$

**Proof:**  $[D_j] \in KK_\Gamma(C_0(TW_j), \mathbb{C})$  denotes the element of  $KK_\Gamma(C_0(TW_j), \mathbb{C})$  given by the Dirac operator of  $TW_j$ . As above we have

$[f] \in KK_\Gamma^0(C_0(TW_1), C_0(TW_2))$ . The Kasparov product pairing

$$KK_\Gamma^0(C_0(TW_1), C_0(TW_2)) \otimes_{\mathbb{Z}} KK_\Gamma^0(C_0(TW_2), \mathbb{C}) \rightarrow KK_\Gamma(C_0(TW_1), \mathbb{C})$$

has  $[f] \otimes_{C_0(TW_2)} [D_2] = [D_1] \quad \square$

Due to (1.19), (1.15) applies to determine a homomorphism

$$\mu : K^i(\cdot, \Gamma) \rightarrow K_i[C_r^*\Gamma]$$

(1.20) **Conjecture Z.** For any (discrete) group

$\Gamma$   $\mu : K^i(\cdot, \Gamma) \rightarrow K_i[C_r^*\Gamma]$  is an isomorphism of abelian groups ( $i = 0, 1$ ).

**Remarks:** From a slightly heuristic point of view, conjecture (1.20) can be

viewed as asserting that any element of  $K_0[C_r^*\Gamma]$  is the index of a

$\Gamma$ -equivariant elliptic operator on a proper  $C^\infty$   $\Gamma$ -manifold. The only

relations imposed on these indices to obtain  $K_0[C_r^*\Gamma]$ , according to the conjecture, are the "obvious" index-preserving relations on the symbols.

Conjecture (1.20) appears to be quite a strong statement. Its truth implies validity of:

- (i) Novikov conjecture on homotopy invariance of higher signatures [9].
- (ii) Gromov - Lawson - Rosenberg conjecture on topological obstructions to the existence of Riemannian metrics of positive scalar curvature [15] [16] [33].
- (iii) Kadison - Kaplansky conjecture that for  $\Gamma$  torsion-free there are no projections in  $C_r^*\Gamma$  other than 0 and 1 .

If  $\Gamma$  is torsion-free, then  $K^i[\cdot, \Gamma] = K_i(B\Gamma)$ , the  $i$ -th K-homology (with compact supports) of the classifying space  $B\Gamma$  .

### §2. Chern Character

Let  $S(\Gamma)$  be the set of all elements in  $\Gamma$  of finite order. The identity element of  $\Gamma$  is in  $S(\Gamma)$  .

$$(2.1) \quad S(\Gamma) = \{ \gamma \in \Gamma \mid \gamma \text{ is of finite order} \}$$

$\Gamma$  acts on  $S(\Gamma)$  by conjugation.  $F\Gamma$  denotes the permutation module (with coefficients  $\mathbb{C}$ ) so obtained. Thus a typical element of  $F\Gamma$  is a finite formal sum

$$\sum_{\gamma \in S(\Gamma)} \lambda_\gamma [\gamma] \quad \lambda_\gamma \in \mathbb{C}$$

In the standard way  $F\Gamma$  is a vector space over  $\mathbb{C}$  . The (right) action of  $\Gamma$  on  $F\Gamma$  is

$$\left( \sum_{\gamma \in S(\Gamma)} \lambda_\gamma [\gamma] \right) \alpha = \sum_{\gamma \in S(\Gamma)} \lambda_\gamma [\alpha^{-1} \gamma \alpha]$$

$\alpha \in \Gamma$   
 $\lambda_\gamma \in \mathbb{C}$

$H_j(\Gamma, F\Gamma)$  denotes the  $j$ -th homology group of  $\Gamma$  with coefficients  $F\Gamma$ . Let

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$\Gamma$  act on  $\mathbb{C}$  trivially. Then

$$(2.2) \quad H_j(\Gamma, F\Gamma) = \text{Tor}_{\mathbb{C}(\Gamma)}^j(F\Gamma, \mathbb{C}) \quad j = 0, 1, 2, \dots$$

In (2.3)  $\mathbb{C}(\Gamma)$  is the group algebra of all finite formal sums

$$\sum_{\gamma \in S(\Gamma)} \lambda_\gamma [\gamma].$$

Set 
$$H_{\text{ev}}(\Gamma, F\Gamma) = \bigoplus_j H_{2j}(\Gamma, F\Gamma)$$

$$H_{\text{odd}}(\Gamma, F\Gamma) = \bigoplus_j H_{2j+1}(\Gamma, F\Gamma)$$

There is then [6] a Chern character

$$\text{ch} : K^0(\cdot, \Gamma) \rightarrow H_{\text{ev}}(\Gamma, F\Gamma)$$

$$\text{ch} : K^1(\cdot, \Gamma) \rightarrow H_{\text{odd}}(\Gamma, F\Gamma)$$

constructed by using cyclic cohomology. Its crucial property is that it becomes an isomorphism after tensoring  $K^i(\cdot, \Gamma)$  with  $\mathbb{C}$ . Hence

$$\text{ch} : K^0(\cdot, \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow H_{\text{ev}}(\Gamma, F\Gamma)$$

$$\text{ch} : K^1(\cdot, \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow H_{\text{odd}}(\Gamma, F\Gamma)$$

are isomorphisms of vector spaces over  $\mathbb{C}$ .

Consider the diagram

$$\begin{array}{ccc} K^i(X, \Gamma) & \xrightarrow{\mu} & K_0^* \left[ \begin{smallmatrix} \mathbb{C} \\ \Gamma \end{smallmatrix} \right] \\ \downarrow \text{ch} & & \downarrow \\ H^{\text{ev}}(\Gamma, F\Gamma) & \longrightarrow & K_0 \left[ \begin{smallmatrix} \mathbb{C} \\ \Gamma \end{smallmatrix} \right] \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

in which the right vertical arrow is the tautological map. The lower horizontal arrow is defined by requiring commutativity in the diagram, and is denoted



$$\mu : H_{ev}(\Gamma, F\Gamma) \rightarrow K_0[C_r^* \Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

The same procedure determines

$$\mu : H_{odd}(\Gamma, F\Gamma) \rightarrow K_1[C_r^* \Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

(2.4) *Conjecture C*. For any group  $\Gamma$

$$\mu : H_{ev}(\Gamma, F\Gamma) \rightarrow K_0[C_r^* \Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

and

$$\mu : H_{odd}(\Gamma, F\Gamma) \rightarrow K_1[C_r^* \Gamma] \otimes_{\mathbb{Z}} \mathbb{C}$$

are isomorphisms of vector spaces over  $\mathbb{C}$ .

**Remarks:** If (1.20) is valid for a group  $\Gamma$  then so is (2.4). The point of (2.4) is that it relates homological invariants of  $\Gamma$  to the K theory of  $C_r^* \Gamma$ .

Let  $L = \{\gamma_1, \gamma_2, \gamma_3, \dots\}$  be a subset of  $S(\Gamma)$  such that any element in  $S(\Gamma)$  is conjugate to one and only one of the  $\gamma_i$ .  $Z(\gamma_i)$  denotes the centralizer of  $\gamma_i$  in  $\Gamma$ . Let  $H_j(Z(\gamma_i), \mathbb{C})$  be the  $j$ -th homology of  $Z(\gamma_i)$  with coefficients  $\mathbb{C}$  and trivial action of  $Z(\gamma_i)$  on  $\mathbb{C}$ . Then for each  $j=0, 1, 2, \dots$

$$(2.5) \quad H_j(\Gamma, F\Gamma) = \bigoplus_i H_j(Z(\gamma_i), \mathbb{C})$$

(2.5) is useful in calculating  $H_*(\Gamma, F\Gamma)$  in examples.

### §3. Finite Groups and Abelian Groups

As noted by J. Rosenberg, operator algebraists tend to think that any statement involving only finite or abelian groups is trivial. This is surely debatable, but let us check (1.20) and (2.4) for such groups.

(3.1) **Lemma:** Conjecture C is valid for any finite group.

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**Proof:** Let  $\Gamma$  be a finite group.

$K_1[C_r^*\Gamma] = 0$  and  $H_j(\Gamma, F\Gamma) = 0$  for  $j > 0$ , so (2.4) is valid in the odd case.  $K_0[C_r^*\Gamma] = R(\Gamma)$ , the representation ring of  $\Gamma$ . Let  $Cl(\Gamma)$  be the  $\mathbb{C}$ -vector-space of all complex-valued functions on  $\Gamma$  which are constant on each conjugacy class.

(3.2)  $Cl(\Gamma) = \{f : \Gamma \rightarrow \mathbb{C} \mid f(\gamma) = f(\alpha^{-1}\gamma\alpha) \text{ for all } \gamma, \alpha \in \Gamma\}$   
 Let  $\chi : R(\Gamma) \rightarrow Cl(\Gamma)$  be the map which assigns to a representation its character.

$H^0(\Gamma, F\Gamma) = Cl(\Gamma)$  and  $\chi : R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow Cl(\Gamma)$  is the inverse to

$$\mu : Cl(\Gamma) \rightarrow R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} . \quad \square$$

(3.3) **Lemma.** Conjecture  $\mathbb{Z}$  is valid for any finite group.

**Proof:** Let  $\Gamma$  be a finite group. Any (continuous) action of a finite group is proper. Let  $\Gamma$  act on a point. This is a final object in  $\mathcal{C}(\cdot, \Gamma)$  and it is immediate that  $K^i(\cdot, \Gamma) = K_i[C_r^*\Gamma]$ . □

(3.4) **Lemma.** Conjecture  $\mathbb{C}$  is valid for any abelian group.

**Proof:** Let  $\Gamma$  be an abelian group.  $\hat{\Gamma}$  denotes the Pontryagin dual of  $\Gamma$ . Forget the group structure on  $\hat{\Gamma}$ , and view  $\hat{\Gamma}$  as a compact Hausdorff topological space. Then  $C_r^*\Gamma = C(\hat{\Gamma})$  and  $K_i[C_r^*\Gamma] = K^i(\hat{\Gamma})$  the K-theory of  $\hat{\Gamma}$  as defined topologically by Atiyah and Hirzebruch [1]. Moreover  $H_j(\Gamma, F\Gamma) = H^j(\hat{\Gamma}; \mathbb{C})$  where  $H^j(\hat{\Gamma}; \mathbb{C})$  is the j-th Čech cohomology group