Chapter one

INTRODUCTION

1 Introduction

Waves occur throughout Nature in an astonishing diversity of physical, chemical and biological systems. During the late nineteenth and the early twentieth century, the linear theory of wave motion was developed to a high degree of sophistication, particularly in acoustics, elasticity and hydrodynamics. Much of this 'classical' theory is expounded in the famous treatises of Rayleigh (1896), Love (1927) and Lamb (1932).

The classical theory concerns situations which, under suitable simplifying assumptions, reduce to linear partial differential equations, usually the wave equation or Laplace's equation, together with linear boundary conditions. Then, the principle of superposition of solutions permits fruitful employment of Fourier-series and integral-transform techniques; also, for Laplace's equation, the added power of complex-variable methods is available.

Since the governing equations and boundary conditions of mechanical systems are rarely strictly linear and those of fluid mechanics and elasticity almost never so, the linearized approximation restricts attention to sufficiently small displacements from some known state of equilibrium or steady motion. Precisely how small these displacements must be depends on circumstances. Gravity waves in deep water need only have wave-slopes small compared with unity; but shallow-water waves and waves in shear flows must meet other, more stringent, requirements. Violation of these requirements forces abandonment of the powerful and attractive mathematical machinery of linear analysis, which has reaped such rich harvests. Yet, even during the nineteenth century, considerable progress was made in understanding aspects of weakly-nonlinear wave propagation, the most notable theoretical accomplishments being those of Rayleigh in acoustics and Stokes for water waves.

Throughout the present century, development of the linear theory of wave motion in fluids and of hydrodynamic stability has been steady and substantial: much of this is described in the books of Lin (1955), Stoker (1957), Chandrasekhar (1961), Lighthill (1978) and Drazin & Reid (1981). In contrast, the present vigorous interest in nonlinear waves and stability in fluids dates mainly from the 1960s. Particularly deserving of mention are the monographs of Eckhaus (1965), Whitham (1974), Phillips (1977) and Joseph (1976) and the collections edited by Leibovich & Seebass (1974) and Swinney & Gollub (1981). Related works by Weiland & Wilhelmsson (1977) on waves in plasmas and Nayfeh (1973) on perturbation methods are also of interest to fluid dynamicists.

The great scope, and even greater volume, of recent work on nonlinear waves and stability pose a daunting task for any student entering the field and a continuing, time-consuming challenge to all who try to keep abreast of recent developments. Comprehensive, yet broad, surveys of research in this area become increasingly difficult to write as the subject expands. But collections of more narrowly-focused reviews by groups of specialists often fail to emphasize the many similarities which exist between related areas; similarities which can reveal fresh insights and generate new ideas.

The underlying theme of the present work is that of wave interactions, primarily in incompressible fluid dynamics. But similar mathematical problems arise in a variety of other disciplines, especially plasma physics, optics, electronics and population dynamics: accordingly, some of the work cited derives from the latter fields of study.

Many fascinating and unexpected wave-related phenomena occur in fluids. For instance, water-wave theory has experienced a revolution in the last two decades: solutions are now available, for waves modulated in space as well as time, which exhibit properties as diverse as solitons, side-band modulations, resonant excitation, higher-order instabilities and wave-breaking. Recent progress has been no less dramatic in nonlinear hydrodynamic stability: the role of mode interactions in the processes leading towards fully-developed turbulence in shear flows is now fairly well understood, and the discovery of low-dimensional 'chaos' in certain fluid flows and in corresponding differential equations is of great current interest. Throughout the history of mathematical analysis, fluid mechanics has provided a challenge and source of inspiration for new theoretical developments: there is every indication that this situation will persist for generations to come.

Chapter 2 is devoted to linear wave interactions, but the remainder of this work concerns aspects of nonlinearity. The underlying assumptions are usually those necessary for development of a *weakly* nonlinear theory: that is to say, linear theory is considered to provide a good starting point in the search for better, higher-order, approximations. However, the nonlinear evolution equations which result from such approximations are sometimes amenable to exact solution: when this is so, an account of their properties is given.

Nonlinear problems are treated in broad categories, on the basis of mathematical rather than physical similarity. Chapter 3 provides a general theoretical introduction; then Chapter 4 treats wave-driven mean flows and waves modified by weak mean flows. Chapter 5 deals with cases of three-wave resonance driven by nonlinearities which are quadratic in wave amplitudes; Chapter 6 concerns nonlinear evolution of a single dominant wave-mode which experiences cubic nonlinearities and Chapter 7 mainly considers interaction of several (typically three or four) wave-modes coupled by cubic nonlinearities. Chapter 8 briefly considers local secondary instabilities and aspects of turbulence. Included in most categories are problems concerning surface waves, internal waves in stratified or rotating fluids and wave-modes in thermal convection and shear flows. Inviscid, and so in some sense conservative, systems are treated side by side with dissipative ones, in order to demonstrate similarities and differences. Typically, the resulting nonlinear evolution equations are soluble analytically in conservative cases, but have rarely been solved other than numerically in dissipative ones. Numerical work which attempts to encompass high-order nonlinearities beyond the range of present analytical techniques is discussed where appropriate.

The use of non-rigorous, sometimes non-rational, procedures – most notably series truncation – is a feature of much work of undoubted interest and value. Unlike Joseph (1976), I have not scrupled to give a full account of the 'state of the art': but it must firmly be borne in mind that the connection between a theoretical model so derived and physical reality is often unclear and perhaps less close than the original author's enthusiasm led him to believe. It is also true that many of the physical configurations so readily envisaged by theoreticians can be rather intractable for experimentalists: even the most obvious restriction to channels of finite length, width and depth immediately causes difficulties! The tendency to make comparisons between theories and experiments which are not strictly comparable is natural and widespread. Theories which are rationally deduced, for some limiting case, have restricted domains of validity which may not overlap with available experimental evidence: comparisons made outwith this range of validity are no more rational – indeed may be less so – than those based on less rigorous theories. Throughout this work, the existing experimental evidence is discussed.

Mechanical systems normally vibrate when displacements from equilibrium are resisted by restoring forces. Examples in fluid mechanics are sound waves, surface gravity and capillary waves, and internal waves sustained by density-stratification, uniform rotation or electromagnetic fields. Such waves may exist in fluid otherwise at rest and they are usually damped by diffusive processes associated with viscosity, thermal or electromagnetic conductivity. But doubly or triply diffusive systems are known to support other instabilities, such as 'salt fingering'.

Relative motion of parts of the fluid, maintained by moving boundaries or applied stresses, modifies wave properties and admits new, possibly unstable, modes. The (Kelvin–Helmholtz) instability of waves at a velocity discontinuity and the centrifugal (Rayleigh–Taylor) instability of differentially rotating flows were among the first to be successfully analysed by linear theory. In unstable rotating flows, the centrifugal force is analogous to the destabilizing body force due to buoyancy in fluid layers heated from below: the latter causes convective (Bénard) instability.

Surface tension provides a restoring force on plane surface waves; but it causes instability of cylindrical columns or jets of liquid. This occurs for geometrical reasons related to the total curvature of the deformed surface, and is analogous to certain instabilities of magnetic flux tubes. Variations in surface tension, due to gradients of temperature or concentration of adsorbed contaminants, may also enhance or inhibit instabilities.

The linear instability of parallel and nearly-parallel flows in channels, boundary layers, unbounded jets and wakes is profoundly influenced by the presence of one or more 'critical layers' where the local flow velocity is close to the phase velocity of a wavelike perturbation. When the primary velocity profile has no inflection point, there are no unstable inviscid modes. But viscosity plays a dual role: as well as providing dissipation, it can also admit new unstable modes which continually absorb energy from the primary flow at the critical layer. Such viscous instability has similarities with Landau damping of plasmas.

Density stratification and the presence of boundaries also play dual roles. A gravitationally-stable density distribution may suppress shear-flow instability; but it can also admit new modes which may interact linearly or nonlinearly to give instability. Likewise, a boundary may enhance viscous dissipation, largely due to the intense oscillatory boundary layer in its vicinity; but it can also reflect wave energy generated elsewhere within the flow and so encourage wave growth.

These few examples serve to illustrate the variety and subtlety of instability mechanisms in fluids. Excellent detailed accounts of linear stability theory are presently available, which it is pointless to duplicate here. The existence of linear instability of a particular flow indicates that this flow cannot normally persist, but will evolve into another type of motion if given an arbitrary small disturbance. However, it is sometimes possible to stabilize a flow by eliminating potentially unstable modes: the party trick of inverting a gauze-covered glass of water is an example, for the gauze prevents growth of the longer wavelength gravitationally-unstable modes not already stabilized by surface tension. Of more practical interest are recent attempts to suppress boundary-layer instability by artificially creating a wave with phase such as to 'cancel' the spontaneously-growing mode. Such stabilization by controlled vibration is effective in dynamical systems with just a few degrees of freedom – for instance the inverted pendulum – but may also induce new parametric instabilities.

If interest is restricted to a finite region of space, say the surface of an aeroplane wing or turbine blade, the mere existence of instability is not the only important aspect. One needs to know whether a disturbance of certain size initiated at some location, say part of the leading edge, will attain significant amplitudes within the region of interest; and, if so, where the greatest amplitudes will occur. Hence, consideration of spatial, as well as temporal, growth is important.

Though linear theory may successfully yield criteria for onset of instability to small disturbances (and sometimes may not!) a finite disturbance can assume a form remote from that of the most unstable linear mode. It *may* happen that nonlinear effects stabilize the disturbance at some small fixed amplitude and that its form broadly resembles the single linear mode from which it evolved.

An instance of this is the toroidal-vortex motion in Taylor–Couette flow between concentric rotating cylinders, at Taylor numbers marginally above the critical one for onset of linear instability. Other examples are near-critical Bénard convection and wind-generated ripples in rather shallow water at just above the critical wind speed. In all such cases, there is a stable solution of the nonlinear equations in the immediate vicinity of the critical conditions for onset of linear instability: this solution bifurcates at the critical point from the trivial zero-amplitude solution.

But, when nonlinear terms have a destabilizing influence, there is no *stable* small-amplitude solution near the critical point and large enough disturbances typically evolve to more complex states. As one moves further from the linear critical conditions, even those constant-amplitude solutions which were stable may lose their stability and support spontaneous growth of other modes. In a similar way, water waves, which are neutrally-stable according to linear theory, exhibit nonlinear instability and modulation.

When a flow becomes very irregular, it is normally described as being turbulent. In fully-developed turbulence, there is no discernible regularity of spatial or temporal structure: Fourier spectra in both space and time are then continuous and broadband, without distinct peaks. When not fully developed, turbulence may be intermittent, confined to localized regions which propagate within an otherwise laminar (though disturbed) flow. A weaker sort of turbulence is found in certain flows which retain a dominant periodic structure amid the broadband 'noise': an example is Taylor–Couette flow at very large Taylor numbers, where spatially-periodic toroidal vortices persist.

Still weaker apparently chaotic motions may occur due to the mutual interaction of a small number of modes: though the temporal structure may be broadband, usually with a few dominant peaks, the spatial structure remains highly organized. Behaviour of this kind, indicative of a 'strange attractor' in the solution space of the governing equations, has deservedly received much recent attention. Both Bénard convection and Taylor–Couette flow can exhibit such behaviour. However, frequent use of the word 'turbulence' in this connection seems misplaced: although the motion is certainly 'chaotic' in time, it remains highly organized in space.

Sometimes, instability and subsequent nonlinear growth have no connection whatever with turbulence. The capillary instability of liquid jets leads to breaking into discrete droplets, usually of regular size; other interfacial instabilities also lead to droplet formation and entrainment. Low Reynolds-number flow of thin liquid films, down an incline under gravity or horizontally under an airflow, may support large-amplitude but still periodic waves or may break up to form dry patches.

Throughout most of this work, the governing equations are the incompressible Navier–Stokes equations,

$$\left.\begin{aligned}(\partial/\partial t+\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\rho_0^{-1}\,\nabla p+\mathbf{f}+\nu\,\nabla^2\mathbf{u},\\ \nabla\cdot\mathbf{u} &= 0.\end{aligned}\right\} \qquad (1.1\text{a, b})$$

Here, $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ respectively denote the velocity vector and pressure at each point $\mathbf{x}$ and instant t and $\mathbf{f}$ is a body force per unit mass. The fluid density ρ_0 is taken to be constant, though this constant may differ in different fluid layers; also, continuous changes in density, assumed small compared with ρ_0, may be incorporated into the gravitational body force (the so-called Boussinesq approximation). The kinematic viscosity ν is also assumed constant and is related to the dynamic viscosity coefficient μ by $\nu = \mu/\rho_0$. Equation (1.1a) yields three scalar momentum equations, one for each co-ordinate direction, and (1.1b) is the continuity equation.

Equations (1.1) are frequently expressed in dimensionless form, relative to characteristic scales of mass, length and time. If the latter are defined by a length L, velocity V and the density ρ_0, dimensionless counterparts of (1.1) are

$$\left.\begin{aligned}(\partial/\partial T + \mathbf{U}\cdot\nabla_1)\mathbf{U} &= -\nabla_1 P + \mathbf{F} + R^{-1}\nabla_1^2\mathbf{U},\\ \nabla_1\cdot\mathbf{U} &= 0,\end{aligned}\right\} \tag{1.1a, b$'$}$$

with the new variables related to the old by $\mathbf{U} = \mathbf{u}/V$, $P = p/\rho_0 V^2$, $\mathbf{F} = \mathbf{f}L/V^2$. The new space co-ordinates, if Cartesian, and dimensionless time T are respectively

$$(X, Y, Z) = (x/L, y/L, z/L), \quad \nabla_1 \equiv (\partial/\partial X, \partial/\partial Y, \partial/\partial Z), \quad T = tV/L.$$

Viscosity is now represented by the *Reynolds number* $R \equiv VL/\nu$. In the following chapters, lower-case symbols are sometimes used to denote these dimensionless variables: there should be no risk of confusion.

The choice of scales for non-dimensionalization is to some extent arbitrary, but strong conventions exist. For example, plane Poiseuille flow through a plane channel is usually characterized by the half-width of the channel and the maximum flow velocity at mid-channel, yielding the dimensionless velocity profile

$$U(Z) = 1 - Z^2 \quad (-1 \leqslant Z \leqslant 1). \tag{1.2}$$

Similarly, boundary-layer flows may be non-dimensionalized relative to the (local) free-stream velocity and displacement thickness.

When there occur variations of temperature θ, and so of density, (1.1) must be supplemented by the thermal equation and by an equation of state expressing variation of density with θ. In the Boussinesq approximation, the former becomes

$$(\partial/\partial t + \mathbf{u}\cdot\nabla)\theta = \kappa\nabla^2\theta \tag{1.3}$$

where κ is thermal diffusivity, and consequent density variations from ρ_0 are considered sufficiently small to be retained only in the gravitational body force $\rho\mathbf{g}$ per unit volume. The dimensionless counterpart of (1.3) has κ replaced by $Pr^{-1}R^{-1}$ where $Pr \equiv \nu/\kappa$ is the *Prandtl number*.

A steady state $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$, $p = p_0(\mathbf{x})$ which satisfies (1.1) may experience a perturbation to

$$\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}'(\mathbf{x}, t), \quad p = p_0 + \epsilon p'(\mathbf{x}, t),$$

where ϵ is a small parameter characteristic of the initial magnitude of the perturbation. From (1.1),

$$\left.\begin{aligned}(\partial/\partial t+\mathbf{u}_0\cdot\nabla)\,\mathbf{u}'+(\mathbf{u}'\cdot\nabla)\,\mathbf{u}_0 &= -\rho_0^{-1}\nabla p'+\mathbf{f}'+\nu\nabla^2\mathbf{u}'-\epsilon(\mathbf{u}'\cdot\nabla)\,\mathbf{u}',\\ \nabla\cdot\mathbf{u}' &= 0,\end{aligned}\right\}\quad(1.4\text{a, b})$$

where $\epsilon\mathbf{f}'$ denotes any perturbation of the body force from its steady-state value. When the disturbance is sufficiently small, it may be justifiable to neglect the term $\epsilon(\mathbf{u}'\cdot\nabla)\,\mathbf{u}'$ in (1.4a): if so, the resultant set of equations for the disturbance is linear and may be solved to find a first approximation to the true perturbed solution. Weakly-nonlinear theory then builds on this by constructing the solution as a series in ascending powers of ϵ.

When viscosity is negligible, equations (1.1) reduce to Euler's equations. If the body force $\mathbf{f}$ is conservative (say $\mathbf{f}=-\nabla\Omega$), these greatly simplify for irrotational flows: for then the vorticity $\nabla\times\mathbf{u}$ remains zero at all times if zero initially. Accordingly, the velocity is expressible as $\mathbf{u}=\nabla\phi$ in terms of a scalar velocity potential $\phi(\mathbf{x},t)$ and (1.1b) immediately yields Laplace's equation. Integration of (1.1a) along any line element within the fluid gives

$$\left.\begin{aligned}\partial\phi/\partial t+p/\rho_0+\Omega+\tfrac{1}{2}(\nabla\phi)^2 &= f(t),\\ \nabla^2\phi &= 0\end{aligned}\right\}\quad(1.5\text{a, b})$$

and the arbitrary function $f(t)$ may be absorbed into ϕ without loss. Here, the nonlinear Euler's equations have reduced exactly to the linear Laplace's equation, without restriction on any disturbance amplitude, and p is given directly by (1.5a) once ϕ is known. However, in many cases to be discussed, the boundary conditions remain nonlinear and so solution is not straightforward.

The physical condition at solid boundaries is that the velocity of the fluid immediately adjacent to the boundary equals that of the boundary: i.e. $\mathbf{u}(\mathbf{x},t)=\mathbf{u}_b(\mathbf{x},t)$ on the boundary surface $B(\mathbf{x},t)=0$. Here, $\mathbf{u}_b$ denotes the velocity of material particles of the boundary. The boundary itself must satisfy a kinematic condition connecting $\mathbf{u}_b$ with the boundary position $B=0$. However, for inviscid flows, the 'no-slip' boundary condition must be discarded and only the velocity component normal to the boundary is prescribed: i.e. $(\mathbf{u}-\mathbf{u}_b)\cdot\hat{\mathbf{n}}=0$ where $\hat{\mathbf{n}}$ is the unit normal to the boundary.

At free surfaces and fluid interfaces, there are both kinematic and dynamical boundary conditions. Continuity of velocity (or, for inviscid flows, the normal component of velocity) across interfaces is required; also the location of the interface is related to the velocity of particles comprising it by a kinematic condition. In addition, dynamical boundary conditions express the force balance at the interface. In Cartesian form, the stress tensor σ_{ij} $(i, j = 1, 2, 3)$ within either fluid (designated by superscripts 1, 2) and the unit normal $\hat{\mathbf{n}} = \hat{n}_j$ at the interface satisfy

$$(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})\,\hat{n}_j = T_i \quad (i = 1, 2, 3) \tag{1.6}$$

with summation over j. Here, $T_i = \mathbf{T}$ represents interfacial forces per unit area; when these derive solely from surface tension, $\mathbf{T}$ equals $\gamma(\nabla \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}}$ where γ is the coefficient of interfacial surface tension. The stress tensor σ_{ij} is related to $\mathbf{u} = u_i$ and p by

$$\sigma_{ij} = -p\delta_{ij} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \tag{1.7}$$

where $\mathbf{x} = x_i$ denote Cartesian co-ordinates and δ_{ij} is the Krönecker delta. At a free surface, σ_{ij} is zero for the absent fluid.

Since these boundary conditions apply at the moving interface, the position of which may be unknown, approximations valid for small displacements from some known location are usually employed. The boundary conditions applicable to inviscid water-wave theory are set out in §§11 and 14. Both the kinematic equation and the pressure boundary condition are inherently nonlinear; further nonlinearities result from constructing the approximate boundary conditions at the mean level of the water surface.

On nomenclature, note that Figures are numbered by Chapter but equations by section. For instance, Figure 6.1 is in Chapter 6 and equation (6.1) is in §6, Chapter 2.

Chapter two

LINEAR WAVE INTERACTIONS

2 Flows with piecewise-constant density and velocity

2.1 *Stability of an interface*

We begin by considering the flow shown in Figure 2.1. Two inviscid incompressible fluids of effectively unlimited extent have respective constant densities ρ_1, ρ_2 and horizontal velocities U, 0. Their common interface is situated at $z = 0$. Gravitational acceleration g acts downwards, in the $-z$ direction, and there may be an interfacial surface tension γ.

Figure 2.1. Kelvin–Helmholtz flow configuration.