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978-0-521-36791-2 - An Introduction to Hankel Operators
Jonathan R. Partington
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0. INTRODUCTION

In Riemann, Hilbert or in Banach space
Let superscripts and subscripts go their ways.
Our asymptotes no longer out of phase,
We shall encounter, counting, face to face.
Stanislaw Lem (The Cyberiad)

We apologise for the fact that in the title of the Tensors talk in the last newsletter, the words "theoretical physics" came out as "impossible ideas".
Archimedeans' Newsletter, January 1986.

Many have been led astray by their speculations,
And false conjectures have impaired their judgement.
Ecclesiasticus 3, 24.

A *Hankel matrix* is one of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \end{pmatrix}$$

that is, a matrix $\{(c_{ij}): i, j = 0, \ldots, \infty\}$, where c_{ij} depends only on $i+j$, so can be written $c_{ij} = a_{i+j}$, for some sequence $a_0, a_1, a_2, \ldots$

Under suitable conditions such a matrix gives rise in a natural way to a linear map (an operator) Γ on the Hilbert space l_2 of square summable sequences, and we have that

$$(\Gamma x)_i = \Sigma_0^\infty a_{i+j} x_j,$$

for $x = (x_0, x_1, x_2, \ldots) \in l_2$. Γ is a *Hankel operator*.

Similarly, a *Hankel Integral Operator* on $L_2(0, \infty)$ has the representation

$$\Gamma x(t) = \int_0^\infty h(t + s)\, x(s)\, ds,$$

so that the *kernel*, $h(t + s)$, depends on the sum of the two variables involved.

As we shall see in more detail later, l_2 is isomorphic to the Hardy space H_2 of analytic functions on the unit disc $\{|z| < 1\}$: this is the space of all functions $f(z) = \Sigma_0^\infty a_n z^n$ with norm $\|f\|^2 = \Sigma |a_n|^2 < \infty$. We thus have a connection between Hankel operators and complex variable theory, which turns out to be very important. Similarly, $L_2(0, \infty)$ is easily related to

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another Hardy space, this time of functions defined on the right half plane $\mathbf{C}_+$, using the Laplace transform.

Hankel operators have in recent years been shown to have widespread applications to both Systems Theory and Approximation Theory: we explore these here.

In Chapter 1 we start with some general operator theory. Compact operators on Hilbert spaces can be written in the form

$$\Gamma x = \Sigma_1^\infty \sigma_i (x, v_i) w_i,$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$, and (v_i) and (w_i) orthonormal sequences in the given Hilbert space. The σ_i are called *singular values (approximation numbers, generalised eigenvalues)* and have many important properties. For example we can consider what it means to say that $\Sigma \sigma_i < \infty$, or that $\Sigma \sigma_i^2 < \infty$ (*nuclear* operators and *Hilbert-Schmidt* operators).

Hardy spaces are introduced in Chapter 2. For the applications to Hankel operators we are only concerned with H_2, H_∞ and (occasionally) H_1, and we give a more elementary discussion than is customary (for example we are able to avoid the use of maximal functions entirely). We also treat Hardy spaces on $\mathbf{C}_+$ by considering their equivalence with Hardy spaces on the disc.

Having established the background we are able to introduce Hankel operators in Chapter 3. Nehari's Theorem and the Carathéodory-Fejér and Nevanlinna-Pick problems are treated. In addition we establish Hartman's theorem on compact Hankel operators.

Hankel integral operators on $L_2(0, \infty)$ and their equivalent forms on $H_2(\mathbf{C}_+)$ are discussed in Chapter 4. Most results here are obtained using equivalences with Hankel operators on the disc.

An elementary treatment of linear systems and H_∞ is presented in Chapter 5. Some infinite-dimensional systems (where the associated Hankel operator is of infinite rank) are discussed. Here we give the physical motivation for Model Reduction – approximation by simpler functions in suitable norms.

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In Chapter 6 we present Beurling's Theorem and the Adamjan-Arov-Krein results on Hankel-norm approximation. Here we follow Power's simplified treatment, giving additional proofs, examples and explanations of this rather deep problem.

The final chapter connects the general operator theory of the first chapter with the Hardy space theory. Various results on Hilbert-Schmidt and nuclear Hankel operators are presented, culminating in the recent results of Peller, Coifman and Rochberg, Bonsall and Walsh, and including various inequalities which give L_1 and H_∞ error bounds for model reduction.

We conclude with an appendix covering various background results in functional analysis which may be unfamiliar to some readers. These include standard results from Operator Theory and Measure Theory, and we give them in their simplest form.

These notes are based on those for a Part III Mathematics course on Hankel operators given to an audience of Mathematicians and Engineers at Cambridge University in the Michaelmas Term, 1987. I am grateful to Dr B. Bollobás, Dr T.K. Carne, Dr K. Glover, Dr T.W. Körner, Mr D.C. McFarlane and Dr R. Ober for useful discussions and comments; also to the Departments of Engineering and of Pure Mathematics and Mathematical Statistics of Cambridge University, to Fitzwilliam College, Cambridge, and to the Science and Engineering Research Council for their assistance.

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1. COMPACT OPERATORS ON A HILBERT SPACE

In this first chapter, we begin by considering linear operators in general – we specialise to Hankel operators in Chapter 3. Although it is possible to discuss operators defined on a general normed space, we shall not do so, but just consider linear operators defined on a complete inner-product space, a Hilbert space. The properties in which we are interested are of greatest importance when the operator is compact, that is, close to being a finite-rank operator (a formal definition will be given later).

For compact operators which are also Hermitian there is the Spectral Theorem, which shows how the action of the operator is fully determined by its eigenvalues and eigenvectors. From this we move to the Schmidt expansion of a general compact operator, and come naturally to the definition of the approximation numbers (singular values) of a compact operator (to be denoted (σ_i)).

A brief discussion of the polar decomposition follows: this enables us to refer to the modulus of an operator, itself an operator with several useful properties.

We spend the remainder of the chapter in considering operators of the class C_p $(1 \le p < \infty)$, that is with $\Sigma_1^\infty \sigma_i^p < \infty$. These form normed spaces, of which the spaces C_1 of nuclear operators, and C_2 of Hilbert-Schmidt operators are by far the most important. We establish the fact that these are indeed normed spaces (in fact C_2 is an inner-product space), discussing the trace function en route.

The results of this chapter are mostly standard and can be found, in various forms, in the cited books of Dunford and Schwartz, Gohberg and Krein, and Schatten.

All operators will be assumed to be continuous unless otherwise specified, and the underlying Hilbert spaces will be complex.

For $A: H \to H$ linear, we recall that its norm is given by $\|A\| = \sup \{\|Ax\|/\|x\|,\ x \neq 0\}$. Also the *adjoint* A^* of A satisfies $(A^*x, y) = (x, Ay)$ for all x and y, and $A^{**} = A$. A is a *compact* operator if the closure of $A(U)$ is compact, where U denotes the unit ball $\{\|x\| \le 1\}$ of H. An equivalent, simpler condition for compactness for operators on a Hilbert space is that A is

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compact if and only if there is a sequence (A_n) of finite rank operators (i.e. $A_n(H)$ is a finite-dimensional subspace), such that $\|A_n - A\| \to 0$. Equivalently again, A is compact if and only if, given any bounded sequence (x_n), the sequence (Ax_n) has a convergent subsequence.

If A is a compact operator, then so are A^* and any scalar multiple of A. The sum of two compact operators is compact, and if A is compact and B is bounded then both AB and BA are compact.

We recall the Spectral Theorem for Compact Hermitian Operators on a Hilbert Space (see the Appendix for more details):

Proposition 1.1 *A is a compact Hermitian operator if and only if there exists a sequence (λ_n) of real numbers which tends to zero, and an orthonormal basis (x_n) of H, such that*

$$Ax = \Sigma_1^\infty \lambda_n(x, x_n)x_n.$$

The (x_n) are eigenvectors of A and the (λ_n) eigenvalues. With respect to this orthonormal basis, A thus has a diagonal matrix.

We observe that there will always be a certain non-uniqueness in this representation: the (x_n) can always be multiplied by scalars of modulus one, and, in the case when we have repeated eigenvalues (eigenspaces of (finite) dimension greater than 1) we can choose an orthonormal basis for such an eigenspace arbitrarily.

We note that the operators A_m $(m = 1, 2, \ldots)$ defined by

$$A_m x = \Sigma_1^m \lambda_n(x, x_n)x_n$$

satisfy rank $(A_m) < \infty$ and $\|A_m - A\| \to 0$, so that A is indeed the norm limit of finite rank operators.

We say that A is *positive*, and write $A \geq 0$, if A is compact Hermitian with positive eigenvalues. It then has a unique positive *square root*, $A^{1/2}$, defined by

$$A^{1/2}x = \Sigma_1^\infty \lambda_n^{1/2}(x, x_n)x_n,$$

which satisfies $A^{1/2}A^{1/2} = A$. For the uniqueness, we note that if $B \geq 0$ and $B^2 = A$, then the eigenvectors of B are eigenvectors of A, and thus if $Bx = \Sigma_1^\infty \mu_n(x, y_n)y_n$, then

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$Ax = \Sigma_1^\infty \mu_n^2(x, y_n)y_n$. Now, looking at the eigenspaces of A, we obtain the uniqueness of the square root.

Theorem 1.2 (Schmidt expansion of a compact operator) *An operator T is compact if and only if it there exist orthonormal sequences (v_i), (w_i), $i \geq 1$, and scalars σ_i decreasing to 0, such that*

$$Tx = \Sigma_1^\infty \sigma_i(x, v_i)w_i.$$

Proof 'If': Writing $T_m x = \Sigma_1^m \sigma_i(x, v_i)w_i$, we have rank$(T_m) \leq m$ and $\|T_m - T\| = \sigma_{m+1} \to 0$.

'Only if': Since $(T^*Tx, x) = (Tx, Tx) \geq 0$ and $(T^*Tx, y) = (x, T^*Ty)$, we have $T^*T \geq 0$. Let $\lambda_1, \lambda_2, \ldots$ be the nonzero eigenvalues of T^*T, ordered in decreasing size, $v_1, v_2, \ldots$ the corresponding eigenvectors (orthonormal), and $\sigma_i = \lambda_i^{1/2}$. Now write $w_i = Tv_i/\sigma_i$. We thus have $(w_i, w_j) = (Tv_i, Tv_j)/\sigma_i\sigma_j = (T^*Tv_i, v_j)/\sigma_i\sigma_j = \sigma_i(v_i, v_j)/\sigma_j = \delta_{ij}$, i.e. the (w_i) are orthonormal.

Note that $T^*Tx = 0$ if and only if $Tx = 0$, so that

$$Tx = \Sigma_1^\infty (x, v_i)Tv_i = \Sigma_1^\infty \sigma_i(x, v_i)w_i.$$

Also $Tv_i = \sigma_i w_i$ and $T^*w_i = \sigma_i v_i$; $TT^*w_i = \sigma_i^2 w_i = \lambda_i w_i$, and $T^*x = \Sigma_1^\infty \sigma_i(x, w_i)v_i$.

The numbers (σ_i) are called *singular values* (sometimes *approximation numbers*, *s-numbers* or *generalised eigenvalues.)*

Corollary 1.3 *If A is an m-by-m matrix, we can find unitary matrices U and V and a positive semi-definite diagonal matrix D such that $A = UDV$.*

Proof A corresponds to a finite rank operator $T: \mathbf{C}^m \to \mathbf{C}^m$. With respect to the orthonormal bases (v_i) and (w_i) (extended if necessary by adding vectors from the kernels of T and T^*), T has the diagonal matrix D. Changing back to the standard orthonormal basis transforms D into UDV, where U and V are unitary matrices.

We can interpret $\sigma_1(T)$ as $\|T\|$. More generally we have the following result.

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Theorem 1.4 *For $n \geq 1$,*

$$\sigma_n(T) = \inf \{\|T - S\| : \text{rank}(S) < n\}.$$

The infimum is actually attained.

Proof We may assume without loss of generality that n is at least 2, since for $n = 1$, $S = 0$ will do. Clearly, taking

$$Sx = \Sigma_1^{n-1} \sigma_i(x, v_i)w_i,$$

we have rank(S) < n and

$$(T - S)(x) = \Sigma_n^{\infty} \sigma_i(x, v_i)w_i,$$

and so $\|T - S\| = \sigma_n$.

Suppose now that R is any operator of rank k, say, and consider L, the linear span of the vectors $v_1, \ldots, v_{k+1}$. Since $\dim(L) > \text{rank}(R)$, we see that the restriction $R: L \to \text{Im } R$ is not injective and there exists a vector x of norm 1 with $x \in L$ and $Rx = 0$. But $\|Tx\| \geq \sigma_{k+1}\|x\|$, since the coordinates of x are each magnified at least that much, and so $\|(T - R)x\| \geq \sigma_{k+1}\|x\|$, which implies that $\|T - R\| \geq \sigma_{k+1} \geq \sigma_n$, and the result follows.

This explains why the σ_i are sometimes called *approximation numbers* of T. When T is not compact, but merely bounded, we can still define

$$\sigma_i(T) = \inf \{\|T - S\| : \text{rank}(S) < i\},$$

and clearly $\sigma_i(T) \to 0$ if and only if T is compact.

Corollary 1.5 $\sigma_{m+n-1}(S + T) \leq \sigma_m(S) + \sigma_n(T)$ *and* $\sigma_{m+n-1}(ST) \leq \sigma_m(S)\sigma_n(T)$ *for* $m, n \geq 1$. *In particular,* $\sigma_m(ST) \leq \sigma_m(S)\|T\|$ *and* $\sigma_m(TS) \leq \sigma_m(S)\|T\|$.

Proof For any $\varepsilon > 0$ we may choose operators S_1 of rank at most $m-1$, and T_1 of rank at most $n-1$, with $\|S - S_1\| \leq \sigma_m(S) + \varepsilon$ and $\|T - T_1\| \leq \sigma_n(T) + \varepsilon$.
Then $\|S + T - (S_1 + T_1)\| \leq \sigma_m(S) + \sigma_n(T) + 2\varepsilon$, and $\text{rank}(S_1 + T_1) \leq m + n - 2$, which means that the left hand side is at least $\sigma_{m+n-1}(S + T)$. The result follows on letting $\varepsilon \to 0$. Similarly

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$$\|ST - (S_1(T - T_1) + ST_1)\| \le \|(S - S_1)(T - T_1)\| \le (\sigma_m(S) + \varepsilon)(\sigma_n(T) + \varepsilon),$$

which implies the second inequality, since the rank of $S_1(T - T_1) + ST_1$ is at most $m + n - 2$.

The Polar Decomposition of a compact operator

Suppose, as usual, that

$$Tx = \Sigma_1^\infty \sigma_i(x, v_i)w_i,$$

so that

$$T^*Tx = \Sigma_1^\infty \sigma_i^2(x, v_i)v_i.$$

We define the *modulus of T*, $|T|$, to be the operator $(T^*T)^{1/2}$, that is

$$|T|x = \Sigma_1^\infty \sigma_i(x, v_i)v_i,$$

and we define the operator $U = U_T$ by

$$Ux = \Sigma_1^\infty (x, v_i)w_i,$$

i.e. $U(v_i) = w_i$ and U is an isometry (not in general compact) of the closed linear span of $\{v_1, v_2, ...\}$ onto the closed linear span of $\{w_1, w_2, ...\}$, which takes its orthogonal complement to zero. This is a *partial isometry*.

The *polar decomposition* of T is then $T = U_T|T|$, which may be compared with the writing of a nonzero complex number as $(z/|z|).|z|$ (or, for $z = 0$, just 0 times 0!)

Proposition 1.6 *Let $T = U_T|T|$ be the polar decomposition of a compact operator T. Then $|T| = U_T^*T$. Moreover, if also $T^* = U_{T^*}|T^*|$, then $U_T^* = U_T$ and hence $T = |T^*|U_T$.*

Proof This is merely a matter of checking what each of these operators actually does.

Since $U_Tx = \Sigma_1^\infty (x, v_i)w_i$, thus mapping v_i to w_i and anything orthogonal to the v_i to 0, it is simple to verify that $U_T^*x = \Sigma_1^\infty (x, w_i)v_i$, which therefore maps w_i to v_i in a similar fashion. Thus $U_T^*T = U_T^*U_T|T| = |T|$. Moreover $T^* = U_{T^*}|T^*|$, which implies that $T = (T^*)^* = |T^*|U_{T^*}^* = |T^*|U_T$, taking adjoints. This proves the proposition.

In fact the polar decomposition does exist for any bounded operator: $T = U_T|T|$, where $|T|^2 = T^*T$, U_T is a partial isometry with Ker U_T = Ker $|T|$, and $|T| \ge 0$, although we shall

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not prove this.

Definition *We say that a compact operator T is in the class C_p $(1 \le p < \infty)$ if and only if $\Sigma_1^\infty\ \sigma_i(T)^p < \infty$.*

Two particular values of p are of interest (as usual):

C_1: The *nuclear* or *trace-class* operators,

and

C_2: The *Hilbert-Schmidt* operators.

Proposition 1.7 *The class C_p is a linear space and an ideal, that is*

i) $T \in C_p,\ \lambda \in \mathbf{C} \Rightarrow \lambda T \in C_p$;

ii) $S,\ T \in C_p \Rightarrow S + T \in C_p$;

iii) S bounded, $T \in C_p \Rightarrow ST \in C_p$ and $TS \in C_p$.

Proof Since $\sigma_i(\lambda T) = |\lambda|\sigma(T)$ we get (i) immediately.

By Corollary 1.5, $\sigma_{2i-1}(S + T) \le \sigma_i(S) + \sigma_i(T) \le 2\ \max\ (\sigma_i(S),\ \sigma_i(T))$, and hence $(\sigma_{2i-1}(S + T))^p \le 2^p(\sigma_i(S)^p + \sigma_i(T)^p)$, and likewise $(\sigma_{2i}(S + T))^p \le 2^p(\sigma_i(S)^p + \sigma_{i+1}(T)^p)$.

Summing over i we see that

$$\Sigma_1^\infty\ (\sigma_i(S + T))^p \le 2^p(2\Sigma_1^\infty\ \sigma_i(S)^p + 2\Sigma_1^\infty\ \sigma_i(T)^p) < \infty.$$

Finally, $\sigma_i(ST) \le \sigma_i(T)\|S\|$ and $\sigma_i(TS) \le \sigma_i(T)\|S\|$, also by Corollary 1.5, and the result follows.

In fact C_p is a Banach space, with norm $\|T\|_{C_p} = (\Sigma_1^\infty\ \sigma_i(T)^p)^{1/p}$, having properties similar to those of l_p. We shall investigate the properties of C_1 and C_2 in detail. We remark also that $T \in C_p$ if and only if $T^* \in C_p$, simply because T and T^* have the same singular values.

Proposition 1.8 *If $\Sigma_1^\infty\ \sigma_i^2(T) < \infty$, then, for any orthonormal basis (x_i) we have*

$$\Sigma_1^\infty\ \|Tx_i\|^2 = \Sigma_1^\infty\ \sigma_i^2(T).$$

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Proof Let (x_i) and (y_i) be any orthonormal bases. Then

$$\Sigma_i \; \|Tx_i\|^2 = \Sigma_i \; \Sigma_j \; |(Tx_i, \; y_j)|^2,$$

by the Riesz-Fischer theorem (see the Appendix),

$$= \Sigma_{i,j} \; |(x_i, \; T^* y_j)|^2 = \Sigma_{i,j} \; |(T^* y_j, \; x_i)|^2,$$
$$= \Sigma_j \; \|T^* y_j\|^2,$$

again by the Riesz-Fischer theorem.

Hence the answer we get is independent of our choice of orthonormal basis (and we get the same answer for T^* instead of T.) Thus, extending (v_i) to an orthonormal basis, by adding in vectors from the kernel of $T = \Sigma_1^\infty \; \sigma_i(x, \; v_i)w_i$ if necessary, we obtain $\Sigma_1^\infty \; \|Tv_i\|^2$, which is equal to $\Sigma_1^\infty \; \sigma_i^2(T)$, as required.

Corollary 1.9 *The space C_2 is an inner-product space, under the inner product*

$$<S, \; T> = \Sigma_1^\infty \; (Sx_k, \; Tx_k),$$

where (x_k) is an orthonormal basis, and the value obtained is independent of the basis (x_k). Hence the space C_2 becomes a normed space with norm

$$\|S\|_{HS} = (\Sigma_1^\infty \; \|Sx_k\|^2)^{1/2} = (\Sigma_1^\infty \; \sigma_k(S)^2)^{1/2}.$$

Proof Observe first that $\Sigma_1^\infty \; |(Sx_k, \; Tx_k)| \le \Sigma_1^\infty \; \|Sx_k\|.\|Tx_k\|$

$\le (\Sigma_1^\infty \; \|Sx_k\|^2)^{1/2}(\Sigma_1^\infty \; \|Tx_k\|^2)^{1/2} < \infty$, (Cauchy-Schwarz) so that the definition makes sense. Clearly the expression $<S, T>$ will be linear in S, conjugate-linear in T, anti-symmetric and positive definite. We need to show that the formula does not depend on our choice of orthonormal basis (x_k): so we note that $<S, S>$ and $<T, T>$ etc. *are* uniquely determined, and hence, by the polarization identity

$<S, \; T> = \frac{1}{4} \; (<S + T, \; S + T> - <S - T, \; S - T> + i<S + iT, \; S + iT>$
$- \; i<S - iT, \; S - iT>)$, we have the uniqueness of $<S, \; T>$, as required.

We can now say something useful about nuclear operators.

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