Cambridge Studies in Advanced Mathematics 8

Commutative ring theory

Already published

1 W.M.L. Holcombe *Algebraic automata theory*
2 K. Petersen *Ergodic theory*
3 P.T. Johnstone *Stone spaces*
4 W.H. Schikhof *Ultrametric calculus*
5 J.-P. Kahane *Some random series of functions, 2nd edition*
6 H. Cohn *Introduction to the construction of class fields*
7 J. Lambek & P.J. Scott *Introduction to higher-order categorical logic*
8 H. Matsumura *Commutative ring theory*
9 C.B. Thomas *Characteristic classes and the cohomology of finite groups*
10 M. Aschbacher *Finite group theory*
11 J.L. Alperin *Local representation theory*
12 P. Koosis *The logarithmic integral I*
13 A. Pietsch *Eigenvalues and s-numbers*
14 S.J. Patterson *An introduction to the theory of the Riemann zeta-function*
15 H.J. Baues *Algebraic homotopy*
16 V.S. Varadarajan *Introduction to harmonic analysis on semisimple Lie groups*
17 W. Dicks & M. Dunwoody *Groups acting on graphs*
18 L.J. Corwin & F.P. Greenleaf *Representations of nilpotent Lie groups and their applications*
19 R. Fritsch & R. Piccinini *Cellular structures in topology*
20 H Klingen *Introductory lectures on Siegel modular forms*
22 M.J. Collins *Representations and characters of finite groups*
24 H. Kunita *Stochastic flows and stochastic differential equations*
25 P. Wojtaszczyk *Banach spaces for analysts*
26 J.E. Gilbert & M.A.M. Murray *Clifford algebras and Dirac operators in harmonic analysis*
27 A. Frohlich & M.J. Taylor *Algebraic number theory*
28 K. Goebel & W.A. Kirk *Topics in metric fixed point theory*
29 J.F. Humphreys *Reflection groups and Coxeter groups*
30 D.J. Benson *Representations and cohomology I*
31 D.J. Benson *Representations and cohomology II*
32 C. Allday & V. Puppe *Cohomological methods in transformation groups*
33 C. Soulé et al *Lectures on Arakelov geometry*
34 A. Ambrosetti & G. Prodi *A primer of nonlinear analysis*
35 J. Palis & F. Takens *Hyperbolicity, stability and chaos at homoclinic bifurcations*
37 Y. Meyer *Wavelets and operators I*
38 C. Weibel *An introduction to homological algebra*
39 W. Bruns & J. Herzog *Cohen-Macaulay rings*
40 V. Snaith *Explicit Brauer induction*
41 G. Laumon *Cohomology of Drinfield modular varieties I*
42 E.B. Davies *Spectral theory and differential operators*
43 J. Diestel, H. Jarchow & A. Tonge *Absolutely summing operators*
44 P. Mattila *Geometry of sets and measures in Euclidean spaces*
45 R. Pinsky *Positive harmonic functions and diffusion*
46 G. Tenenbaum *Introduction to analytic and probabilistic number theory*
47 C. Peskine *An algebraic introduction to complex projective geometry I*
50 I. Porteous *Clifford algebras and the classical groups*
51 M. Audin *Spinning Tops*
54 J. Le Potier *Lectures on Vector bundles*

Commutative ring theory

HIDEYUKI MATSUMURA
Department of Mathematics, Faculty of Sciences
Nagoya University, Nagoya, Japan

Translated by M. Reid

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521367646

First published in English by Cambridge University Press 1986 as
Commutative ring theory
First paperback edition (with corrections) 1989
Ninth printing 2006

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Matsumura, Hideyuki, 1930–
Commutative ring theory.
Translation of: Kakan kan ron.
Includes index.
1. Cummutative rings. I. Title.
QA251.3.M37213 1986 512′.4 86-11691

ISBN 978-0-521-36764-6 paperback

Transferred to digital printing 2008

Contents

Preface

In publishing this English edition I have tried to make a rather extensive revision. Most of the mistakes and insufficiencies in the original edition have, I hope, been corrected, and some theorems have been improved. Some topics have been added in the form of Appendices to individual sections. Only Appendices A, B and C are from the original. The final section, §33, of the original edition was entitled 'Kunz' Theorems' and did not substantially differ from a section in the second edition of my previous book Commutative Algebra (Benjamin, 2nd edn 1980), so I have replaced it by the present §33. The bibliography at the end of the book has been considerably enlarged, although it is obviously impossible to do justice to all of the ever-increasing literature.

Dr Miles Reid has done excellent work of translation. He also pointed out some errors and proposed some improvements. Through his efforts this new edition has become, I believe, more readable than the original. To him, and to the staff of Cambridge University Press and Kyoritsu Shuppan Co., Tokyo, who cooperated to make the publication of this English edition possible, I express here my heartfelt gratitude.

Hideyuki Matsumura
Nagoya

Introduction

In addition to being a beautiful and deep theory in its own right, commutative ring theory is important as a foundation for algebraic geometry and complex analytic geometry. Let us start with a historical survey of its development.

The most basic commutative rings are the ring $\mathbb{Z}$ of rational integers, and the polynomial rings over a field. $\mathbb{Z}$ is a principal ideal ring, and so is too simple to be ring-theoretically very interesting, but it was in the course of studying its extensions, the rings of integers of algebraic number fields, that Dedekind first introduced the notion of an ideal in the 1870s. For it was realised that only when prime ideals are used in place of prime numbers do we obtain the natural generalisation of the number theory of $\mathbb{Z}$.

Meanwhile, in the second half of the 19th century, polynomial rings gradually came to be studied both from the point of view of algebraic geometry and of invariant theory. In his famous papers of the 1890s on invariants, Hilbert proved that ideals in polynomial rings are finitely generated, as well as other fundamental theorems. After the turn of the present century had seen the deep researches of Lasker and Macaulay on primary decomposition of polynomial ideals came the advent of the age of abstract algebra. A forerunner of the abstract treatment of commutative ring theory was the Japanese Shōzō Sono (On congruences, I–IV, *Mem. Coll. Sci. Kyoto*, **2** (1917), **3** (1918–19)); in particular he succeeded in giving an axiomatic characterisation of Dedekind rings. Shortly after this Emmy Noether discovered that primary decomposition of ideals is a consequence of the ascending chain condition (1921), and gave a different system of axioms for Dedekind rings (1927), in work which was to have a decisive influence on the direction of subsequent development of commutative ring theory. The central position occupied by Noetherian rings in commutative ring theory became evident from her work.

However, the credit for raising abstract commutative ring theory to a substantial branch of science belongs in the first instance to Krull (1899–1970). In the 1920s and 30s he established the dimension theory of Noetherian rings, introduced the methods of localisation and completion, and the notion of a regular local ring, and went beyond the framework of Noetherian rings to create the theory of general valuation rings and Krull rings. The contribution of Akizuki in the 1930s was also considerable; in particular, a counter-example, which he obtained after a year's hard struggle, of an integral domain whose integral closure is not finite as a module was to become the model for many subsequent counter-examples.

In the 1940s Krull's theory was applied to algebraic geometry by Chevalley and Zariski, with remarkable success. Zariski applied general valuation theory to the resolution of singularities and the theory of birational transformations, and used the notion of regular local ring to give an algebraic formulation of the theory of simple (non-singular) point of a variety. Chevalley initiated the theory of multiplicities of local rings, and applied it to the computation of intersection multiplicities of varieties. Meanwhile, Zariski's student I.S. Cohen proved the structure theorem for complete local rings [1], underlining the importance of completion.

The 1950s opened with the profound work of Zariski on the problem of whether the completion of a normal local ring remains normal (Sur la normalité analytique des variétés normales, *Ann. Inst. Fourier* **2** (1950)), taking Noetherian ring theory from general theory deeper into precise structure theorems. Multiplicity theory was given new foundations by Samuel and Nagata, and became one of the powerful tools in the theory of local rings. Nagata, who was the most outstanding research worker of the 1950s, also created the theory of Hensel rings, constructed examples of non-catenary Noetherian rings and counter-examples to Hilbert's 14th problem, and initiated the theory of Nagata rings (which he called pseudo-geometric rings). Y. Mori carried out a deep study of the integral closure of Noetherian integral domains.

However, in contrast to Nagata and Mori's work following the Krull tradition, there was at the same time a new and completely different movement, the introduction of homological algebra into commutative ring theory by Auslander and Buchsbaum in the USA, Northcott and Rees in Britain, and Serre in France, among others. In this direction, the theory of regular sequences and depth appeared, giving a new treatment of Cohen–Macaulay rings, and through the homological characterisation of regular local rings there was dramatic progress in the theory of regular local rings.

The early 1960s saw the publication of Bourbaki's *Algèbre commutative*, which emphasised flatness, and treated primary decomposition from a new angle. However, without doubt, the most characteristic aspect of this decade was the activity of Grothendieck. His scheme theory created a fusion of commutative ring theory and algebraic geometry, and opened up ways of applying geometric methods in ring theory. His local cohomology is an example of this kind of approach, and has become one of the indispensable methods of modern commutative ring theory. He also initiated the theory of Gorenstein rings. In addition, his systematic development, in Chapter IV of EGA, of the study of formal fibres, and the theory of excellent rings arising out of it, can be seen as a continuation and a final conclusion of the work of Zariski and Nagata in the 1950s.

In the 1960s commutative ring theory was to receive another two important gifts from algebraic geometry. Hironaka's great work on the resolution of singularities [1] contained an extremely original piece of work within the ideal theory of local rings, the ring-theoretical significance of which is gradually being understood. The theorem on resolution of singularities has itself recently been used by Rotthaus in the study of excellent rings. Secondly, in 1969 M. Artin proved his famous approximation theorem; roughly speaking, this states that if a system of simultaneous algebraic equations over a Hensel local ring A has a solution in the completion $\hat{A}$, then there exist arbitrarily close solutions in A itself. This theorem has a wide variety of applications both in algebraic geometry and in ring theory. A new homology theory of commutative rings constructed by M. André and Quillen is a further important achievement of the 1960s.

The 1970s was a period of vigorous research in homological directions by many workers. Buchsbaum, Eisenbud, Northcott and others made detailed studies of properties of complexes, while techniques discovered by Peskine and Szpiro [1] and Hochster [H] made ingenious use of the Frobenius map and the Artin approximation theorem. Cohen–Macaulay rings, Gorenstein rings, and most recently Buchsbaum rings have been studied in very concrete ways by Hochster, Stanley, Kei-ichi Watanabe and S. Goto among others. On the other hand, classical ideal theory has shown no sign of dying off, with Ratliff and Rotthaus obtaining extremely deep results.

To give the three top theorems of commutative ring theory in order of importance, I have not much doubt that Krull's dimension theorem (Theorem 13.5) has pride of place. Next perhaps is I.S. Cohen's structure theorem for complete local rings (Theorems 28.3, 29.3 and 29.4). The fact that a complete local ring can be expressed as a quotient of a well-understood ring, the formal power series ring over a field or a discrete valuation ring, is something to feel extremely grateful for. As a third, I would give Serre's characterisation of a regular local ring (Theorem 19.2); this grasps the essence of regular local rings, and is also an important meeting-point of ideal theory and homological algebra.

This book is written as a genuine textbook in commutative algebra, and is as self-contained as possible. It was also the intention to give some thought to the applications to algebraic geometry. However, both for reasons of space and limited ability on the part of the author, we are not able to touch on local cohomology, or on the many subsequent results of the cohomological work of the 1970s. There are readable accounts of these subjects in [G6] and [H], and it would be useful to read these after this book.

This book was originally to have been written by my distinguished friend Professor Masao Narita, but since his tragic early death through illness, I have taken over from him. Professor Narita was an exact contemporary of mine, and had been a close friend ever since we met at the age of 24. Well-respected and popular with all, he was a man of warm character, and it was a sad loss when he was prematurely called to a better place while still in his forties. Believing that, had he written the book, he would have included topics which were characteristic of him, UFDs, Picard groups, and so on, I have used part of his lectures in §20 as a memorial to him. I could wish for nothing better than to present this book to Professor Narita and to hear his criticism.

Hideyuki Matsumura
Nagoya

Conventions and terminology

(1) Some basic definitions are given in Appendixes A–C. The index contains references to all definitions, including those of the appendixes.

(2) In this book, by a *ring* we always understand a commutative ring with unit; ring homomorphisms $A \longrightarrow B$ are assumed to take the unit element of A into the unit element of B. When we say that A is a subring of B it is understood that the unit elements of A and B coincide.

(3) If $f: A \longrightarrow B$ is a ring homomorphism and J is an ideal of B, then $f^{-1}(J)$ is an ideal of B, and we denote this by $A \cap J$; if A is a subring of B and f is the inclusion map then this is the same as the usual set-theoretic notion of intersection. In general this is not true, but confusion does not arise.

Moreover, if I is an ideal of A, we will write IB for the ideal $f(I)B$ of B.

(4) If A is a ring and $a_1, \ldots, a_n$ elements of A, the ideal of A generated by these is written in any of the following ways: $a_1A + a_2A + \cdots + a_nA$, $\sum a_iA$, $(a_1, \ldots, a_n)$ or $(a_1, \ldots, a_n)A$.

(5) The sign $\subset$ is used for inclusion of a subset, including the possibility of equality; in [M] the sign $\subseteq$ was used for this purpose. However, when we say that '$M_1 \subset M_2 \subset \cdots$ is an ascending chain', $M_1 \subsetneq M_2 \subsetneq \cdots$ is intended.

(6) When we say that R is a ring of characteristic p, or write char $R = p$, we always mean that $p > 0$ is a prime number.

(7) In the exercises we generally omit the instruction 'prove that'. Solutions or hints are provided at the end of the book for most of the exercises. Many of the exercises are intended to supplement the material of the main text, so it is advisable at least to glance through them.

(8) The numbering Theorem 7.1 refers to Theorem 1 of §7; within one paragraph we usually just refer to Theorem 1, omitting the section number.