1

Introduction to Normed Algebras; Examples

Introduction

This chapter begins by stating some basic conventions, definitions and notation that will be used throughout the work. Additional standard notations will be introduced from time to time, as needed. The reader should consult the index of notation for reference. Many of the ideas presented in the first section will be familiar to some readers. They are mentioned for the sake of review and to fix our notation. Also, of course, some standard concepts are defined in slightly different ways by different authors, and we wish to make clear our own conventions. The chapter concludes with a number of examples discussed in some depth. We urge readers to acquaint themselves with these since an abstract theory, such as that presented in this work, lacks substance without knowledge of examples.

The first section deals primarily with basic elementary results on normed, semi-normed, or topological linear spaces and algebras. Such topics as ideals, homomorphisms, quotient norms, etc. are discussed, and the role of semi-norms in locally convex topological linear spaces is quickly surveyed. The unitization of an algebra and an important convention about it are also introduced.

In order to enliven the section, we have introduced several interesting or slightly unusual topics which need relatively few prerequisites. Some basic facts about commutant subsets of algebras and maximal commutative subalgebras are presented. For any submultiplicative semi-norm σ on an algebra $\mathcal{A}$, we present the important properties of the non-negative real-valued function on $\mathcal{A}$ defined by

$$\lim_{n\to\infty} \sigma(a^n)^{1/n} \qquad \forall\, a \in \mathcal{A}.$$

Section 1.2 deals with the double centralizer algebra $\mathcal{D}(\mathcal{A})$ of an algebra $\mathcal{A}$. We regard $\mathcal{D}(\mathcal{A})$ as a more natural unitization of a non-unital algebra $\mathcal{A}$. It also allows the classification of extensions of well-behaved Banach algebras under mild hypotheses. This interesting theory is also presented.

Section 1.3 discusses a number of ways in which algebras and Banach algebras can be combined to make new algebras. It introduces direct sums, direct products, subdirect products, both projective and injective limits, ultraproducts and ultrapowers.

Section 1.4 is devoted to the Arens product. This interesting, fundamental and elementary construction, which provides a product on the double dual Banach space of any Banach algebra, is surveyed in some detail and explored more thoroughly in some of the examples which follow.

The remaining six sections present examples. Some of these will probably be familiar but most are discussed in an elementary fashion to show how the ideas arise naturally and to make them accessible even to the beginner. In many years of teaching this material, we have often noted students who become facile with the theory without knowing a reasonable stock of examples. Thus we urge that these sections be read in detail. Algebras of functions are dealt with very briefly since they belong more to the subject matter of Chapter 3. Matrix algebras, operator algebras and group algebras are presented in more detail.

A number of simple examples, which we might have included here are algebras with involutions. Since the second volume of this book will be devoted to a much deeper study of involutions, we omit most of these examples here. For this reason, most algebras of operators on Hilbert space are omitted or slighted even though they rank among the simplest examples, as will be seen when they are presented in Volume II.

1.1 Norms and Semi-norms on Algebras

Sets, Functions and Notation

If $\mathcal{D}$ and $\mathcal{S}$ are sets, we write $\mathcal{D}\backslash\mathcal{S}$ for the *difference set* $\{a \in \mathcal{D} : a \notin S\}$. If f is any function with domain $\mathcal{D}$, and $\mathcal{S}$ is any set, $f|_S$ denotes the *restriction of* f to the domain $\mathcal{D}\cap\mathcal{S}$. In neither case do we insist that S be a subset of $\mathcal{D}$. We sometimes use the notation $f^{\leftarrow}$ for the relation which is the *inverse* of a function f, particularly when $f^{\leftarrow}$ is not a function. (However if f is a function in an algebra $\mathcal{A}$ of linear functions under composition, we always use f^{-1} for the inverse when it is a function in $\mathcal{A}$. Conversely, if f is a function with values in a group in which multiplicative notation is used or in the invertible elements in some ring, and if f belongs to a group or ring of functions in which multiplication is defined pointwise, then f^{-1} will always represent the function defined by $f^{-1}(x) = f(x)^{-1}$ for each x in the domain of f.) If $\mathcal{F}$ is a set of functions, each with domain including $\mathcal{X}$, we write $\mathcal{F}(\mathcal{X})$ for the set $\{f(x) : f \in \mathcal{F},\ x \in \mathcal{X}\}$.

In any topological space we denote the *boundary, closure* and *interior* of a subset $\mathcal{S}$ by $\partial\mathcal{S}$, $\mathcal{S}^-$, and $\mathcal{S}^\circ$, respectively. The *support* supp$f$ of a real- or complex-valued function f defined on a topological space Ω is the closure of the set where f is non-zero.

We will use $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^\bullet$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}^0$ and $\emptyset$ to denote, respectively, the set of *complex numbers,* the *unit disc* $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, the 1-*torus*

$\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, the set of *real numbers*, the set of *non-negative real numbers*, the set of *positive real numbers*, the set of all *integers*, the set $\{1, 2, 3, \ldots\}$ of *natural numbers* or *positive integers*, the set $\{0, 1, 2, \ldots\}$ of *non-negative integers* and the *empty set*. We endow each of these sets with all its usual structure so that, for instance, $\mathbb{R}$ is an ordered normed field. *Open* and *closed intervals* are denoted by $]\cdot,\cdot[$ and $[\cdot,\cdot]$, respectively. The *supremum* of a set in $\mathbb{R}$ is ∞ if the set is unbounded and $-\infty$ if the set is empty. Similar conventions hold for the *infimum*. The *complex conjugate* of a complex number λ will be denoted by λ^*. We frequently use the *Kronecker delta*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The indices i and j may be any type of mathematical object.

It is often convenient to state several definitions or results in a parallel fashion. We do so by listing the various choices involved in order, enclosed in angular brackets and separated by slashes: $\langle \ldots / \ldots / \ldots \rangle$. References within this work are given by a three part number made up of the chapter number, section number and result or subsection number. References to other books and articles are given by mentioning the author's name in the text and then giving the year of publication enclosed in square brackets. Full details are located in the bibliography.

Algebras and Subalgebras

1.1.1 Definition An *algebra* $\mathcal{A}$ over a field $\mathbb{F}$ is a ring which is also a linear space over $\mathbb{F}$ under the same addition and satisfies

$$(\lambda a)b = \lambda(ab) = a(\lambda b) \qquad \forall\ \lambda \in \mathbb{F};\ a, b \in \mathcal{A}.$$

N. B. Throughout this work all linear spaces or algebras will have the **complex** field as their field of scalars unless the contrary is explicitly stated. Occasionally $\mathbb{F}$ will denote either the real or the complex field: $\mathbb{R}$ or $\mathbb{C}$.

An algebra is said to be *unital* if it has a multiplicative identity (*i.e.*, an element 1 satisfying $1a = a1 = a$ for all $a \in \mathcal{A}$).

A *subalgebra* is a subring which is also a linear subspace (*i.e.* it is a linear subspace which contains the product of any of its elements). A subalgebra of a unital algebra which contains the multiplicative identity element of the larger algebra is called a *unital subalgebra*.

We will always state definitions and results for algebras even when the theory for arbitrary rings is no more complicated. When feasible, we will try to state definitions, propositions and arguments so that they also apply to algebras with the real numbers as scalar field but this is not always possible. Note that a subalgebra may be a unital algebra (because it contains its own identity element) without being a unital subalgebra (because the larger algebra is either nonunital or has a different identity element).

An element a in an algebra $\mathcal{A}$ is called an $\langle$ *idempotent* / *nilpotent* $\rangle$ if it satisfies $\langle\ a^2 = a\ /\ a^n = 0$ for some $n \in \mathbb{N}\ \rangle$. An idempotent is said to be *proper* if it is not a multiplicative identity element for the algebra to which it belongs. A non-zero proper idempotent is said to be *nontrivial*. Finally, two idempotents e and f are *orthogonal* if they satisfy $ef = fe = 0$.

An element a in an algebra $\mathcal{A}$ is called a $\langle$ *left* / *right* $\rangle$ *divisor of zero* if there is some non-zero $b \in \mathcal{A}$ satisfying $\langle\ ab = 0\ /\ ba = 0\ \rangle$. An element which is either a left or right divisor of zero is called a *divisor of zero* and an element which is both is called a *two-sided divisor of zero* and a *joint divisor of zero* if the same element b can be used on both sides.

Historical Remarks on Algebras

The term "algebra", was first applied, in the sense used in this work, by Benjamin Peirce [1870]. His interesting paper, which was published posthumously in 1881, was in part a philosophical discussion intended to establish the modern 20th century view that "Mathematics is the science which draws necessary conclusions." As an example, he defined "linear associative algebras", by axioms and derived a number of consequences. For another third of a century most mathematicians studied algebras as "systems of hypercomplex numbers". This term denoted a linear space with a given distinguished basis and a multiplication table for the basis elements giving each product as a specified linear combination of basis elements. Shortly after the turn of the century, under the influence of Leonard Eugene Dickson, the term "algebra", and the axiomatic definition and viewpoint were generally adopted by mathematicians. For instance, at Dickson's suggestion the term "algebra", (but not the axiomatic definition) is used in the text of Joseph H. Maclagan Wedderburn's classic paper [1907] establishing his decomposition theorems. Dickson used both the term "algebra", and the axiomatic definition in his influential book [1927]. See also Karen Hunger Parshall [1985].

Linear Spans, Convex Hulls, Products and Sums of Subsets

When $\mathcal{S}$ is a subset of a linear space $\mathcal{X}$ we write $\operatorname{span}(\mathcal{S})$ for the linear span of $\mathcal{S}$ and $\operatorname{co}(\mathcal{S})$ for the convex hull of $\mathcal{S}$. If $\mathcal{X}$ is the linear span of a linearly independent set, we call the set a *Hamel basis* for $\mathcal{X}$. A subset $\mathcal{S}$ of a linear space is said to be *balanced* if λx belongs to $\mathcal{S}$ for all complex λ satisfying $|\lambda| \le 1$ and all $x \in \mathcal{S}$. The *balanced convex hull* of a subset $\mathcal{S}$ of $\mathcal{X}$, denoted by $\operatorname{ba}(\mathcal{S})$, is the set

$$\left\{\sum_{j=1}^{n} \lambda_j x_j : n \in \mathbb{N};\ \lambda_j \in \mathbb{C},\ x_j \in \mathcal{S} \text{ for } j = 1, 2, \ldots, n;\ \sum_{j=1}^{n} |\lambda_j| \le 1\right\}.$$

Let $\mathcal{A}$ be an algebra. If $\mathcal{S}$ and $\mathcal{T}$ are linear subspaces of $\mathcal{A}$, we will denote the set $\operatorname{span}\{ab : a \in \mathcal{S},\ b \in \mathcal{T}\}$ by $\mathcal{ST}$. Also if a is an element of $\mathcal{A}$ and $\mathcal{S}$ and

$\mathcal{T}$ are linear subspaces of $\mathcal{A}$, we denote the set span$\{bac : b \in \mathcal{S}, c \in \mathcal{T}\}$ by $\mathcal{S}a\mathcal{T}$. Similar variants are also used including expressions like $(1-a)\mathcal{A} = \{b - ab : b \in \mathcal{A}\}$ and $(\lambda - a)\mathcal{A} = \{\lambda b - ab : b \in \mathcal{A}\}$ even in nonunital algebras. In general, the algebras we study will be noncommutative. Hence it is important to agree that the symbol $\Pi_{j=1}^n a_j$ means $a_1 a_2 \cdots a_n$. For any integer n strictly greater than one and any linear subspace $\mathcal{S}$, we denote the set span$\{\Pi_{j=1}^n a_j : a_j \in \mathcal{S}\}$ by $\mathcal{S}^n$. If $\mathcal{S}$ and $\mathcal{T}$ are any subsets of $\mathcal{A}$, we denote the set $\{a+b : a \in \mathcal{S}, b \in \mathcal{T}\}$ by $\mathcal{S}+\mathcal{T}$. If $\mathcal{S} = \{a\}$ is a singleton, we replace $\mathcal{S}+\mathcal{T}$ by $a+\mathcal{T}$. (These notations were introduced into the theory of algebras by Wedderburn [1907] following similar notation used in group theory by George Frobenius.)

The multiplication of linear subspaces just defined satisfies the associative law and the distributive law relative to addition. The addition of subsets satisfies the commutative and associative laws. We will use these properties without further comment. (The reader should note the contrast with the notation $\mathcal{F}(\mathcal{X}) = \{f(x) : f \in \mathcal{F}; x \in \mathcal{X}\}$, where $\mathcal{F}$ is a set of functions with domains including $\mathcal{X}$. In this case no linear span is taken even if $\mathcal{F}$ and $\mathcal{X}$ are both linear spaces.) The *kernel* of a linear map $\varphi: \mathcal{X} \to \mathcal{Y}$ is the linear subspace $\ker(\varphi) = \{x \in \mathcal{X} : \varphi(x) = 0\}$,

Ideals, Homomorphisms and Quotients of Algebras

We use the word *ideal* to mean a two-sided ideal. Thus a *left ideal* or *right ideal* is not a special kind of ideal. In an algebra, a one-sided ideal or an ideal is a linear subspace since it is a subalgebra. A *homomorphism* between algebras is a ring homomorphism which is also a linear map. Thus it is a linear map φ satisfying

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad \forall\, a, b \in \mathcal{A}$$

if $\mathcal{A}$ is the domain of φ. Its *kernel* is $\{a \in \mathcal{A} : \varphi(a) = 0\}$. If $\mathcal{A}$ and $\mathcal{B}$ are algebras, the set of homomorphisms of $\mathcal{A}$ into $\mathcal{B}$ will be denoted by Hom$(\mathcal{A}, \mathcal{B})$. As usual, a bijective homomorphism of algebras is called an *isomorphism* and an *automorphism* if it maps $\mathcal{A}$ onto itself. If $\mathcal{A}$ is a unital algebra, the simplest automorphisms are the *inner automorphisms* $a \mapsto b^{-1}ab$ where b is invertible and b^{-1} is its inverse. A linear map or homomorphism $\varphi: \mathcal{A} \to \mathcal{B}$ is said to be *injective* if its kernel is $\{0\}$, and *surjective* if each element of $\mathcal{B}$ is the image under φ of some element of $\mathcal{A}$.

A linear map $\varphi: \mathcal{A} \to \mathcal{B}$ which is anti-multiplicative (*i.e.*, $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$) is called an *anti-homomorphism*. "Isomorphic", "anti-isomorphism", "anti-isomorphic", etc. are defined as usual. If $\varphi: \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism (or anti-homomorphism), then its kernel is an ideal of $\mathcal{A}$. Conversely, if $\mathcal{I}$ is an ideal of an algebra $\mathcal{A}$, then the set $\mathcal{A}/\mathcal{I} = \{a + \mathcal{I} \ : \ a \in \mathcal{A}\}$ of cosets has (in the usual way) the structure of an algebra such that the natural map $\varphi: \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a homomorphism with kernel $\mathcal{I}$ (*i.e.*, $(a+\mathcal{I})+(b+\mathcal{I}) = a+b+\mathcal{I}$, $\lambda(a+\mathcal{I}) = \lambda a+\mathcal{I}$, $(a+\mathcal{I})(b+\mathcal{I}) = (ab+\mathcal{I})$) and $\varphi(a) = a+\mathcal{I}$.

If $\mathcal{I}, \mathcal{K}$ and $\mathcal{L}$ are ideals in an algebra $\mathcal{A}$ and $\mathcal{L}$ is included in $\mathcal{I}$, we expect the reader to be familiar with the natural isomorphisms between $\mathcal{I}/(\mathcal{I}\cap\mathcal{K})$ and $(\mathcal{I}+\mathcal{K})/\mathcal{K}$: $a+\mathcal{I}\cap\mathcal{K} \longleftrightarrow a+\mathcal{K}$; and between $\mathcal{A}/\mathcal{I}$ and $(\mathcal{A}/\mathcal{L})/(\mathcal{I}/\mathcal{L})$: $a+\mathcal{I} \longleftrightarrow a+\mathcal{L}+(\mathcal{I}/\mathcal{L})$. They are both easily checked.

Historical Remark on Ideals and Quotients

We wish to comment on the significance of the word "ideal.", In attempting to prove Fermat's last theorem, both Augustin-Louis Cauchy and Ernst Eduard Kummer were led to consider, at least implicitly, algebraic number fields, *i.e.*, finite algebraic extensions of the rational field. At first they each made the mistake of implicitly assuming that the fundamental theorem of arithmetic (the uniqueness of factorization into primes) held. When Peter Gustav Lejeune Dirichlet pointed out the error, Kummer in *ca.* 1845 invented what he called ideal numbers in order to restore unique factorization. Richard Dedekind replaced Kummer's ideal numbers with the special subrings which we now use, and he quite naturally called them ideals as we still do. In the theory of algebras, ideals were considered for several decades under the name "invariant sub-complexes", *cf.* Theodor Molien [1893], Élie Cartan [1898], and Wedderburn [1907]. The term "ideal", was used in the context of algebras by Dickson in his influential book [1927].

The quotient algebra of an algebra modulo an ideal was introduced by Molien [1893]. In the case of certain special commutative algebras, the idea was accepted much earlier and can be traced back to Karl Friedrich Gauss' theory of congruences.

Hereditary Subalgebras and Normal Subalgebras

The two classes of subalgebras mentioned in the heading have some similarity to ideals in that they are defined by requiring that certain products of sets have special properties. A subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$ is said to be *hereditary* if it satisfies $\mathcal{BAB} \subseteq \mathcal{B}$. Obviously the intersection of a left ideal and right ideal has this property. The concept arose to generalize this situation to closed *-subalgebras of C*-algebras which were generated by their positive parts. The term seems to have been first used by Francois Combes in [1969]. This followed the recognition by Reese T. Prosser [1963] that these algebras have the form $\mathcal{L}\cap\mathcal{L}^*$ where $\mathcal{L}$ is a closed left ideal, and a series of papers by Gert K. Pedersen, beginning with [1966], which had the goal of understanding the non-closed case.

The concept of normal subalgebras was introduced by Marc A. Rieffel [1979a], [1979b]. It is motivated by the following consideration which relates to Section 1.9 below. Suppose G is a locally compact group and H is an open subgroup. Then $L^1(H)$ can be regarded as a subalgebra of $L^1(G)$ simply by extending functions on H to be zero on the rest of G. The problem then is to distinguish the subalgebras arising from normal subgroups from those arising from non-normal subgroups. Suppose H is normal. If $\mathcal{I}$ is an ideal of $L^1(G)$, then $\mathcal{K} = \mathcal{I} \cap L^1(H)$ would satisfy $L^1(G)\mathcal{K} = \mathcal{K}L^1(G)$, and this would only be true in general when H is a normal subgroup. Thus Rieffel defined a subalgebra of an algebra to be *normal* if the above invariance property (*i.e.*, for $\mathcal{B}$ a subalgebra of $\mathcal{A}$ and $\mathcal{I}$ an ideal of $\mathcal{A}$, $\mathcal{A}(\mathcal{I} \cap \mathcal{B}) = (\mathcal{I} \cap \mathcal{B})\mathcal{A}$) holds for a suitable class of ideals. In the cited paper he was able to use this concept to generalize certain aspects of the theory of induced representations from normal subgroups to the setting of normal subalgebras of algebras. We mention this concept here in part because it has not yet been exploited despite its obvious potential. With further refinements, possibly involving double centralizer algebras, it may play an important role.

The Reverse Algebra

If $\mathcal{A}$ is an algebra, the *reverse* of $\mathcal{A}$ is the algebra $\mathcal{A}^R$ with the same underlying linear space as $\mathcal{A}$ but with multiplication defined by setting ab in $\mathcal{A}^R$ equal to ba in $\mathcal{A}$. Thus the identity map of $\mathcal{A}$ onto $\mathcal{A}^R$ is an anti-isomorphism, and an algebra $\mathcal{B}$ is anti-isomorphic to $\mathcal{A}$ if and only if it is isomorphic to $\mathcal{A}^R$.

Semi-topological Linear Spaces and Algebras

Much of this work deals with the interplay between algebraic and topological or metric properties. We now introduce the first of these hybrid concepts. *N. B.* Throughout this work, a set $\mathcal{U}$ is called a *neighborhood* of a point x if x is in the interior of $\mathcal{U}$. Thus a neighborhood need not be open.

1.1.2 Definition A *semi-topological linear space* $\mathcal{X}$ is a linear space together with a topology such that the maps

$$\mathbb{C} \times \mathcal{X} \to \mathcal{X} \text{ defined by } (\lambda, x) \mapsto \lambda x \tag{1}$$

$$\mathcal{X} \times \mathcal{X} \to \mathcal{X} \text{ defined by } (x, y) \mapsto x + y \tag{2}$$

are continuous. A *semi-topological algebra* $\mathcal{A}$ is an algebra together with a topology such that the linear space of $\mathcal{A}$ is a semi-topological linear space and the maps of $\mathcal{A}$ into $\mathcal{A}$ defined by

$$a \mapsto ab \text{ and } a \mapsto ba \tag{3}$$

are both continuous for each fixed $b \in \mathcal{A}$. A *topological linear space* or a *topological algebra* is, respectively, a semi-topological linear space or a semi-topological algebra in which the topology is Hausdorff.

Some authors require that the multiplication in a topological algebra $\mathcal{A}$ be jointly continuous from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. In this terminology, a semi-topological algebra is often a Hausdorff topological algebra with singly rather than jointly continuous multiplication.

Separation Properties

Recall that for a semi-topological linear space the T_0, T_1, Hausdorff (T_2) and regular Hausdorff (T_3) separation axioms are equivalent. To see this, consider a neighborhood $\mathcal{U}$ of zero in a topological linear space $\mathcal{X}$. Subtraction is continuous since addition and multiplication by -1 are. Thus there is a neighborhood $\mathcal{V}$ of zero satisfying $\mathcal{V}-\mathcal{V} \subseteq \mathcal{U}$, and hence $\mathcal{V}^- \subseteq \mathcal{U}$. Since translation is a homeomorphism, this shows that for any neighborhood $\mathcal{U}$ of a point in $\mathcal{X}$ there is a neighborhood $\mathcal{V}$ satisfying $\mathcal{V}^- \subseteq \mathcal{U}$. This result shows that T_0 implies T_3. In fact a topological linear space is completely regular since it is uniformizable (*cf. e.g.*, Kelley [1955]). We will say more on this subject when we discuss topological groups. In the next chapter, Section 2.9 will also give more information on topological algebras.

Quotients of Semi-topological Algebras

If $\mathcal{I}$ is an ideal in a semi-topological algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ can be furnished with the quotient topology. We remind the reader that the *quotient topology* defined by the map $\varphi: \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the topology on $\mathcal{A}/\mathcal{I}$ in which a set $\mathcal{U} \subseteq \mathcal{A}/\mathcal{I}$ is open if and only if $\varphi^{\leftarrow}(\mathcal{U})$ is open in $\mathcal{A}$.

1.1.3 Proposition *Let $\mathcal{A}$ be a semi-topological algebra and let $\mathcal{I}$ be an ideal in $\mathcal{A}$.*

(a) *$\mathcal{A}/\mathcal{I}$ is a semi-topological algebra under the quotient topology. It is a topological algebra if and only if $\mathcal{I}$ is closed.*

(b) *If $\mathcal{I}$ is the kernel of a continuous homomorphism into a topological algebra, then $\mathcal{I}$ is closed.*

Proof (a): In order to check that $\mathcal{A}/\mathcal{I}$ is a semi-topological algebra we must check the continuity of the maps $(a+\mathcal{I}, b+\mathcal{I}) \mapsto a+b+\mathcal{I}$, $(\lambda, a+\mathcal{I}) \mapsto \lambda a+\mathcal{I}$, $a+\mathcal{I} \mapsto ab+\mathcal{I}$ and $a+\mathcal{I} \mapsto ba+\mathcal{I}$. We show only the first since the others are similar. Suppose $\mathcal{U}$ is an open neighborhood of $a+b+\mathcal{I}$ in $\mathcal{A}/\mathcal{I}$. By definition of the quotient topology, this means that the preimage $\mathcal{U}'$ of $\mathcal{U}$ under the natural homomorphism $\varphi: \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is open. Thus there are open neighborhoods $\mathcal{V}$ and $\mathcal{W}$ of a and b, respectively, satisfying $\mathcal{V}+\mathcal{W} \subseteq \mathcal{U}'$. Since $\mathcal{U}'$ is a union of cosets of $\mathcal{I}$, we have $(\mathcal{V}+\mathcal{I})+(\mathcal{W}+\mathcal{I}) \subseteq \mathcal{U}'$. However, $\mathcal{V}+\mathcal{I}$ (and similarly $\mathcal{W}+\mathcal{I}$) is open since it is the union of the open sets $\mathcal{V}+a$. Hence $\varphi(\mathcal{V})$ and $\varphi(\mathcal{W})$ are open neighborhoods of $a+\mathcal{I}$ and $b+\mathcal{I}$, respectively, and they satisfy $\varphi(\mathcal{V})+\varphi(\mathcal{W}) \subseteq \mathcal{U}$.

If $\mathcal{I}$ is closed, $\mathcal{A}/\mathcal{I}$ is T_1 and hence a topological algebra. Conversely if $\mathcal{A}/\mathcal{I}$ is T_1, then $\mathcal{I}$ is closed.

(b): If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is continuous and $\mathcal{B}$ is a topological algebra, then the kernel $\mathcal{I} = \varphi^{\leftarrow}(\{0\})$ is closed by the definition of continuity. □

If $\mathcal{A}$ and $\mathcal{B}$ are semi-topological algebras, denote the set of continuous algebra homomorphisms $\varphi\colon \mathcal{A} \to \mathcal{B}$ by $\mathrm{CHom}(\mathcal{A}, \mathcal{B})$. Any $\varphi \in \mathrm{CHom}(\mathcal{A}, \mathcal{B})$ can be factored

$$\mathcal{A} \to \mathcal{A}/\mathcal{I} \to \varphi(\mathcal{A}) \to \mathcal{B} \tag{4}$$

where $\mathcal{I}$ is the kernel of φ. If $\mathcal{B}$ is a topological algebra, then $\mathcal{I}$ is closed so $\mathcal{A}/\mathcal{I}$ and $\varphi(\mathcal{A})$ are topological algebras.

It is easy to see that the closure of a subalgebra, left ideal, right ideal or ideal in a semi-topological algebra is a subset of the same type. A subset $\mathcal{S}$ of an algebra $\mathcal{A}$ is called *commutative* if any two of its elements commute (*i.e.*, if a, $b \in \mathcal{S}$ implies $ab = ba$). It is easy to see that the closure of a commutative subset in a topological algebra is commutative.

Commutants

John von Neumann [1929] introduced both the notation and an appreciation of the importance of commutants. Commutants are sometimes called centralizers, but this word also has other meanings.

1.1.4 Definition If $\mathcal{S}$ is a subset of an algebra $\mathcal{A}$, then the *commutant* of $\mathcal{S}$ is the subset $\mathcal{S}'$ defined by

$$\mathcal{S}' = \{a \in \mathcal{A} : ab = ba \quad \forall\, b \in \mathcal{S}\}.$$

We denote the double commutant $(\mathcal{S}')'$ by $\mathcal{S}''$ and similarly for higher order commutants. The set $\mathcal{A}'$ is called the *center* (German *Zentrum*) of $\mathcal{A}$ and is denoted by $\mathcal{A}_Z$.

1.1.5 Proposition *The commutant $\mathcal{S}'$ of a subset $\mathcal{S}$ of an algebra $\mathcal{A}$ satisfies:*

(a) *$\mathcal{S}'$ is a subalgebra of $\mathcal{A}$.*
(b) *$\mathcal{S} \subseteq \mathcal{S}''$.*
(c) *$\mathcal{S} \subseteq \mathcal{S}'$ if and only if $\mathcal{S}$ is commutative.*
(d) *$\mathcal{S}_1 \subseteq \mathcal{S}_2$ implies $\mathcal{S}_2' \subseteq \mathcal{S}_1'$.*
(e) *$\mathcal{S}' = \mathcal{S}'''$.*
(f) *A subset $\mathcal{C}$ of $\mathcal{A}$ is a maximal (under inclusion order) commutative subset if and only if it satisfies $\mathcal{C} = \mathcal{C}'$. Hence maximal commutative subsets are subalgebras.*
(g) *Every commutative subset $\mathcal{C}$ is included in some maximal commutative subalgebra.*

Proof Most of these results are immediate consequences of the definition. Result (e) comes from applying (d) to (b), and (f) can be seen by noting that $\mathcal{C} \cup \{c\}$ is commutative if $\mathcal{C}$ is and c belongs to $\mathcal{C}'$. Obviously (g) depends on an application of Zorn's lemma. □

We now note the topological consequences of these results. A variant of result (b) was first systematically exploited by Paul Civin and Bertram Yood [1959].

1.1.6 Proposition *Let $\mathcal{A}$ be a topological algebra. For every $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{S}'$ is closed. Hence the center of an algebra and every maximal commutative subalgebra are closed.*

Proof For each $b \in \mathcal{S}$, the set $\{a \in \mathcal{A} : ab - ba = 0\}$ is closed. The intersection of all these sets for $b \in \mathcal{S}$ is just $\mathcal{S}'$. The center is $\mathcal{A}'$ and (f) of the last proposition shows $\mathcal{C} = \mathcal{C}'$. □

For another interesting property of maximal commutative subalgebras refer to Proposition 2.5.3.

Semi-norms

Nearly all of the topological or semi-topological algebras and linear spaces in this work will have topologies defined by a semi-norm or a family of semi-norms. We introduce this simple but important concept now.

1.1.7 Definition A *semi-norm* σ on a linear space $\mathcal{X}$ is a map $\sigma : \mathcal{X} \to \mathbb{R}_+$ satisfying:

$$\text{(a)} \qquad \sigma(x+y) \leq \sigma(x) + \sigma(y) \quad \text{(subadditivity)} \tag{5}$$

$$\text{(b)} \qquad \sigma(\lambda x) = |\lambda|\sigma(x) \quad \text{(absolute homogeneity)} \tag{6}$$

for all $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{C}$. A semi-norm σ on an algebra $\mathcal{A}$ is called an *algebra semi-norm* if it also satisfies:

$$\text{(c)} \qquad \sigma(ab) \leq \sigma(a)\sigma(b) \quad \text{(submultiplicativity)} \tag{7}$$

for all $a, b \in \mathcal{A}$. An algebra semi-norm σ on an algebra $\mathcal{A}$ is called *nontrivial* unless $\mathcal{A}$ has a multiplicative identity at which σ is zero. For any semi-norm σ on a linear space $\mathcal{X}$, define $\mathcal{X}_\sigma$ by

$$\mathcal{X}_\sigma = \{x \in \mathcal{X} : \sigma(x) = 0\}. \tag{8}$$

A semi-norm σ on a linear space $\mathcal{X}$ is called a *norm* if $\mathcal{X}_\sigma = \{0\}$, (*i.e.*, when $\sigma(x) = 0$ implies $x = 0$). A linear space together with a semi-norm or norm is called a *semi-normed* or *normed linear space*. An algebra together with an algebra semi-norm or algebra norm is called a *semi-normed* or *normed algebra*. A normed linear space or normed algebra which is complete in its norm is called a *Banach space* or *Banach algebra*, respectively. A normed linear space is said to be *separable* if it has a countable dense subset. Finally,