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Algebraic representations for complete social networks

Social networks are collections of social or interpersonal relationships linking individuals in a social grouping. The study of social networks has been gaining momentum in the social sciences ever since studies conducted in the 1950s by Barnes (1954), Bott (1957) and others demonstrated the important role of social networks in understanding a number of social phenomena. Social networks have since come to span a diverse theoretical and empirical literature within the social sciences. They have been invoked in a variety of roles in different theoretical contexts and have been conceptualised in a number of ways. The frequency of use of the notion of social network is probably not surprising because an individual's behaviour takes place in the context of an often highly salient network of social relationships. Perhaps more striking is the range of theoretical roles that have been proposed for the social network concept. Social networks have been used to explain various characteristics and behaviours of individuals; they have also been used to account for social processes occurring in both small and large groups of individuals. In addition, they have been viewed as dependent on individual attributes and behaviours, as well as consequences of such aggregate social attributes as the level of urbanisation of a community.

For example, in one type of network research, social scientists have examined the nature of social networks as a function of structural variables such as occupation, stage in life, gender, urbanisation and industrialisation (Blau, 1977; Coates, 1987; Feiring & Coates, 1987; Fischer, 1982; Fischer, Jackson, Stueve, Gerson & McAllister Jones, 1977; Wellman, 1979). In these discussions, the primary focus has been on describing the consequent variation in network characteristics such as the density of ties in a person's local social network, although some authors have expressed the need for more structural concerns (e.g., Friedkin, 1981; Wellman, 1982). Others have considered the interrelationships between social networks and the more traditional sociological categories by examining the distributions of social ties between persons in various categories (e.g., Blau, 1977; Fararo, 1981; Fararo & Skvoretz, 1984; Rytina & Morgan, 1982).

In a second type of network research, a more diverse group of researchers have used social networks as a means of explaining individual behaviour. The classic studies of Barnes (1954) and Bott (1957) fall into this class of network studies, as do many more recent investigations (e.g., Kessler, Price and Wortaman, 1985; Laumann, Marsden & Prensky, 1983). A growing body of work, for example, views psychological characteristics such as mental health as dependent in part on features of an individual's interpersonal environment (Brown & Harris, 1978; Cohen & Syme, 1985; Henderson, Byrne & Duncan-Jones, 1981; Kadushin, 1982; Lin, Dean & Ensel, 1986). The network characteristics selected for study in such investigations have included the density of one's local social network (Kadushin, 1982), the availability of attachment in the network (Brown & Harris, 1978) and its perceived adequacy (Henderson et al., 1981). Wellman (1983, 1988) has summarised the essence of this form of network analysis as its emphasis on structural forms allocating access to scarce resources. Social networks provide both opportunities and constraints for social behaviour and are therefore a necessary part of the background information required to explain behaviour (Campbell, Marsden & Hurlbert, 1986; Granovetter, 1985; Marsden, 1983). A variety of behaviours have been considered in this enterprise, including not only indicators of physical, mental, economic and social well-being (e.g., Campbell et al., 1986; Kadushin, 1982; Kessler & McLeod, 1985; Piliksuk & Froland, 1978) but also such diverse behaviours as option trading (Baker, 1984), more general economic behaviours (Granovetter, 1985) and individual decision-making (Anderson & Jay, 1985; Krackhardt & Porter, 1987).

A third type of network research examines the behaviour of a larger group of individuals as a function of the social networks connecting them. The theme in this work is the assessment of the large-scale, global or "macro" effects of individual or "micro" behaviour constrained by the local network structure. Some empirical illustrations of the approach have been conducted in small- to medium-sized groups against a background of a complete mapping of network links between all members of a specified population of persons. Examples include Sampson's (1969) documentation of a "blow-up" in a monastery and Laumann and Pappi's (1976) description of collective decision-making as a function of network links. In each of these cases, the social behaviour of the group was claimed to be understood in terms of information about the structure of the social relationships of its members.

On a larger scale, more direct mappings of the relationship between network characteristics and population parameters have been attempted.

For example, Granovetter (1974) characterised the local personal networks of a sample of individuals and assessed the relationship between network characteristics and aspects of job-finding. Skog (1986) has argued that network processes may underlie long-term fluctuations in national alcohol consumption rates and has observed the need for a greater understanding of the topology of social networks, that is, of the patterns in which network links are distributed in a population. Many social processes occur as the result of micro interactions among persons connected in a network, and the aggregation of these processes across an entire population can clearly depend on the arrangement of links in the network. Skog argued that the Law of Large Numbers may not hold for certain kinds of network structure, so that the assessment of the impact of network topology has far-reaching significance. Granovetter, also, has stressed the need to take social structure into account for a wide variety of social processes, and his arguments have inspired a good deal of empirical study into the role of network structure in information transmission and other processes (e.g., Friedkin, 1980; Granovetter, 1974, 1982; Lin, Dayton & Greenwald, 1978; Lin & Dumin, 1986; Lin, Ensel & Vaughn, 1981; Murray & Poolman, 1982).

A rather different line of work has examined the mutual dependence of individual and network characteristics in small- to medium-sized groups. For example, Breiger and Ennis (1979) have examined the relationship between individual characteristics and properties of the interpersonal environment in which an individual is located (see also Ennis, 1982). In two case studies, they have established a meaningful set of constraints between individual and network features, so adding to our understanding of how particular people come to hold particular network positions. Oliveri and Reiss (1987) viewed characteristics of an individual's personal network as markers for the individual's social orientation and preferences, and they described the networks of mothers and fathers in a sample of families. Leung, Pattison and Wales (1992) have also investigated the relationship between individual and network characteristics by studying the interdependence of the meaning that an individual ascribes to the word *friend* and the network environment in which the ascription is made. Investigations of these latter kinds may be helpful in preventing personal characteristics of an individual and features of the individual's social network becoming a confusing and imperfect proxy for one another (Hall & Wellman, 1985; Wellman, 1982).

The range of theoretical uses of the concept of social network is therefore broad, but it can be argued that most conceptions of the role of social networks fall into one or both of two main classes. The first includes proposals of some kind of link between properties and/or behaviours of an individual and the immediate or extended network environment in which that individual is located. The second class is characterised by the view that social networks define paths for the flow of social "traffic", so that an understanding of social network structure is essential to an understanding of social processes occurring on that network structure. These two views differ in their explanatory emphasis, and there is no necessary inconsistency between them. For instance, it is reasonable to argue for a mutual interdependence of the characteristics of individuals, groups of individuals and the social relationships that connect them.

Nonetheless, it is perhaps surprising that similar features of social networks tend to be evaluated in a variety of network studies. For example, many of the empirical investigations inspired by these theoretical concerns have described social networks in terms of such characteristics as the size of the network, the density of network ties, the centrality of individuals within the network and their integration into a cohesive unit. Many investigators have relied upon a survey approach, constructing a local network of individuals in the immediate network neighbourhood of a randomly selected individual, whereas others have built a more complete picture of the relationships among all persons in a relatively bounded group.

Some evidence suggests, though, that characteristics of social networks may relate to individual and group behaviour in complex ways and that it may not be sufficient to measure features such as the density, size and centrality of a social network, without regard to other structural characteristics (e.g., Friedkin, 1981; Hall & Wellman, 1985). The argument is particularly cogent where social processes are of interest, that is, where the arrangement of network links has been argued to play a substantial role in the development of the process under study (Granovetter, 1973; Skog, 1986). Thus, in the work reported here, I have presented a representation and a means of analysis for social network data that has some structural complexity. The representation allows a unified approach for both complete and local network data and is intended to be sensitive to the two themes just identified for social network research. It is based on the representations developed by Boorman and White (1976), Mandel (1983), White, Boorman and Breiger (1976) and Winship and Mandel (1983), as well as on various developments of them (including those by Breiger & Pattison, 1986; Pattison, 1982; Pattison, 1989; Pattison & Bartlett, 1982). In this chapter, the case in which network data are obtained for each member of a well-defined group is considered. The forms in which such "complete" network data may arise are reviewed, and then the structural representation that is proposed for them is described. Network data in the form of individual-centred local networks are introduced in chapter 2, together with an analogous form of structural representation.

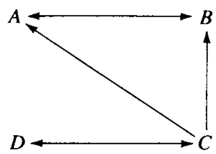


Figure 1.1. A directed graph representation of a friendship network among four members of a work group

Complete network data

The most basic form of social network data can be described as a set of social units, such as individuals, and a collection of pairs of units who are linked by a social relationship of some kind (Freeman, 1989). For example, a Friendship network among a group of individuals belonging to a particular organisation comprises the members of the organisation and the set of pairs of members who are linked by the relation of friendship. An example of such a network for a small, hypothetical work group is shown in Figure 1.1 in the form of a directed graph. Each member of the group is represented as a point, or vertex, of the graph (labelled in Fig. 1.1 by the letters A, B, C and D), and a directed arrow, or edge, links a member to each friend. For example, the link from A to B in Figure 1.1 indicates that A claims B as a friend. The set of group members forms the *vertex* or *node set* X of the graph, and the links defined by pairs of individuals who are friends form the *edge set* of the graph. In Figure 1.1, the vertex set is $X = \{A, B, C, D\}$, and $A \to B$, $B \to A$, $C \to A$, $C \to B$, $C \to D$ and $D \to C$ are the directed edges of the graph. The same network may also be represented in a closely related relational form. The set of organisation members form a set of elements X, and a *relation* F is defined as the set of ordered pairs of members who are linked by a friendship relation. Each ordered pair in the relation F corresponds to a directed edge of the graph of the network. For instance, for the friendship network displayed in Figure 1.1, the relation F may be written as $F = \{(A, B), (B, A), (C, A), (C, B), (C, D), (D, C)\}$.

A third common representation of this kind of network data is a binary matrix. The organisation members again define a set of elements, and these elements may be listed in any order and assigned an integer from 1 to n, where n is the number of members of the group. For instance, A, B, C and D may be assigned the integers 1, 2, 3 and 4, respectively. Then the kth individual in the list may be seen as corresponding to the kth row and the kth column of a square matrix. The cell of the matrix at the intersection of the the ith row and the jth column (i.e., the (i, j)

Table 1.1. *The binary matrix of the friendship network in a small work group*

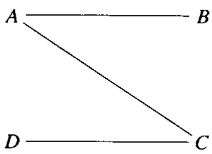
	1	2	3	4
1	0	1	0	0
2	1	0	0	0
3	1	1	0	1
4	0	0	1	0

cell) may be used to record the presence or absence of a friendship link from the ith individual to the jth individual. The cell is usually defined to have an entry of 1 if the relationship of interest is present (i.e., if the ith individual names the jth individual as a friend) and an entry of 0 if the relationship is absent (i.e., if individual i does not name individual j as a friend). The binary matrix corresponding to the friendship network of Figure 1.1 is displayed in Table 1.1. (The matrix is termed *binary* because each of its entries is either zero or one.)

It may be observed that the diagonal entries of Table 1.1 – that is, the entries in cells (1, 1), (2, 2) and so on – are all zero. Correspondingly, there are no links in Figure 1.1 from any vertex to itself. (A link from a vertex to itself in a directed graph is often termed a *loop*.) In this example, an investigator is likely to be interested only in friendship relations between distinct individuals and may not even wish to consider whether it makes sense to speak of an individual being his or her *own* friend. Indeed, in many studies in which network data are generated, it is assumed that the graphs of the networks have no loops, or, equivalently, that the matrices of the networks have zero diagonals.

In some cases, though, loops and non-zero diagonals may possess useful interpretations. If, for instance, the social units of the network are groups of individuals rather than single persons, then it may be meaningful to regard a group as having a friendship relation to itself as well as to other groups. Certain forms of network analysis may also render the use of loops appropriate (e.g., Arabie & Boorman, 1982; Pattison, 1988). In the treatment of network data developed here, it is not assumed that loops are forbidden, even though they do not occur in some of the examples that are presented.

Symmetric and valued networks. A number of variations on this basic account of a social network have been found useful. For instance, in some cases it is reasonable to assume that every link of a network is reciprocated, that is, that if one individual is linked to a second, then the second is also linked to the first. Networks of close friendships may



(a)

$$F = \{(A, B), (A, C), (C, D)\}$$

(b)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

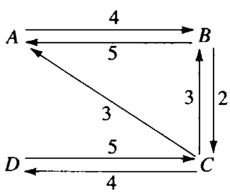
(c)

Figure 1.2. Representations for symmetric network relations: (a) (symmetric) graph; (b) set of unordered pairs; (c) symmetric binary matrix

have this character (e.g., Hammer, 1984). The viability of the assumption is essentially an empirical question, but where it is plausible, the representations described earlier may be simplified to some extent. For a reciprocated network relation, the presentation of the network in diagrammatic form need only indicate the presence of an edge and not its direction. The diagram is then termed a *graph*. Also, the relation corresponding to the network need only indicate the pairs of group members who are related, and the ordering of elements within each pair may be ignored. Finally, the binary matrix of the network may be assumed to be *symmetric*, that is, have the property that the entry in cell (i, j) of the matrix is the same as the entry in cell (j, i). For instance, if i names j as a friend, the assumption of symmetry means that j also names i as a friend. In this case, the entries in the row of the matrix corresponding to an individual are identical to those in the column corresponding to the same individual. These three ways of describing a network having reciprocated links are illustrated in Figure 1.2.

Symmetric relations can also arise from networks whose links are *nondirected* rather than directed *and* reciprocated. For instance, it may be useful in some circumstances to define a (nondirected) link to exist between two individuals in a network if one has contact with the other. The relation may not necessarily be reciprocated, but the direction of the link may be irrelevant for some questions. In fact, many network data have been gathered in this form (e.g., Freeman, 1989). They may be presented in exactly the same way as directed, reciprocated relations, that is, in the form of a collection of unordered pairs, a symmetric binary relation, a graph or a symmetric binary matrix. As a result, they are not distinguished here from reciprocated, directed relations, although it should be noted that, for some purposes, distinction may be advisable (Wasserman & Faust, 1993).

In some other cases, it may be possible to make finer distinctions among network links rather than simply determine their presence or absence (also, Wasserman & Iacobucci, 1986). For instance, the links



(a)

$$\begin{aligned} v(A, B) &= v(C, D) = 4 \\ v(B, A) &= v(D, C) = 5 \\ v(C, A) &= v(C, B) = 3 \\ v(B, C) &= 2 \end{aligned}$$

(b)

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 3 & 3 & 0 & 4 \\ 1 & 1 & 5 & 0 \end{bmatrix}$$

(c)

Figure 1.3. Representations for a valued network relation: (a) valued directed graph; (b) valued relation; (c) valued matrix (The strength of friendship links was assessed on a 5-point scale, with 1 = absent, 2 = a little, 3 = somewhat, 4 = quite strong, 5 = strong. Links of strength 1 have been omitted from the directed graph and valued matrix representations.)

may be measured on a numerical scale, with the scale values indicating the strength of the network link, such as the frequency of contact or the strength of friendship. The nature of the numerical scale will depend on the nature of the measurement procedures used to infer network links and on the properties of the measurements themselves (e.g., Batchelder, 1989). We hope, however, that the scale is at least *ordinal*, that is, it faithfully reflects orderings among network links in terms of strength. Where such numerical information is available, the representations require minor modification, as illustrated in Figure 1.3. The graphical representation takes the form of a *valued graph*, in which each directed or nondirected edge has a numerical value attached. The relational form specifies a mapping v from each (ordered or unordered) pair of elements to a possible value of the network link whereas the matrix representation records the value of the link from node i to node j in the cell of the matrix corresponding to row i and column j. For example, the value of the friendship link from A to B in Figure 1.3 is 4, so that the edge directed from A to B in the graph of the network has value 4; the function v assigns the value 4 to the ordered pair (A, B) (i.e., $V(A, B) = 4$), and the entry in cell $(1, 2)$ of the matrix of the network is 4.

In sum, a single network relation for a specified group of persons may be constructed in any of the four ways implied by our description. That is, it may be a symmetric or nonsymmetric binary relation, or a symmetric or nonsymmetric valued relation.

Multiple networks. In many network studies, more than one type of network relation is of interest, and it is necessary to construct more complex representations. For instance, for the small work group whose

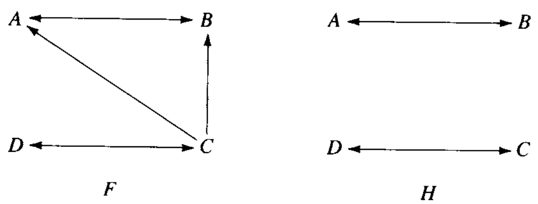


Figure 1.4. A multiple network **W** (F = friendship, H = helping)

Table 1.2. *Binary matrix representation of the multiple network* **W**

Relation	
F	H
0 1 0 0	0 1 0 0
1 0 0 0	1 0 0 0
1 1 0 1	0 0 0 1
0 0 1 0	0 0 1 0

friendship links are displayed in Figure 1.1, information might also be available for a different type of social relationship – for example, who goes to whom for help with work-related problems. This second type of network information is displayed with the first in Figure 1.4 and illustrates a *multirelational* social network. It is a network comprising a single set of network members and more than one type of network relation. In the example of Figure 1.4, two directed graphs are used to present the network. We may also represent the network in terms of two sets of ordered pairs, one set F for friendship links and one set H for helping links. The set F is as before, whereas $H = \{(A, B), (B, A), (C, D), (D, C)\}$. The matrix representation also requires two matrices, one for the friendship relation and one for the helping relation; these are presented in Table 1.2. The multirelational network is labelled **W**, and we may write $\mathbf{W} = \{F, H\}$.

In a representation of this kind, the symbols F and H may actually be used in two distinct ways. As we have just made explicit, each symbol denotes the collection of ordered pairs of elements of X who are linked by a relation of the specified type (either friendship or helping). The symbols will also be used, though, as labels for network links; for instance, we shall say that there is a link of type F from node i to node j if (i, j) is an ordered pair in F. Since the context will always make the intended meaning clear, we shall use the symbols for the relations in both of these ways in what follows.

Table 1.3. *Types of complete network data*

	Single relation		Multiple relations	
	Symmetric	Nonsymmetric	Symmetric	Nonsymmetric
Binary	Symmetric network	Network	Symmetric (multiple) network	(Multiple) network
Valued	Valued symmetric network	Valued network	Valued symmetric (multiple) network	Valued (multiple) network

Each network of a multirelational network may be assessed as a relation that is either symmetric or nonsymmetric, and binary or valued. For simplicity of presentation, we shall describe the overall network in terms of the minimum level of complexity needed to describe each constituent network. Thus, if any of the networks in the multirelational network is valued or nonsymmetric, we present each member of the network in the form appropriate to valued or nonsymmetric relations. For instance, if one relation in a multiple network is binary and another is valued, then we report both relations in valued form. This convention leads to the basic classification of multirelational networks summarised in Table 1.3. The table characterises network data as having either single or multiple relations, and as having constituent networks that are either symmetric or nonsymmetric, and binary or valued. It also identifies the labels to be used for the various forms of network data. For most of the work presented here, the basic form of network data that we shall assume is that of multiple networks, but in chapter 7 we also consider the case of multiple valued networks. The features of these two forms of network data, and the nature of their representations, are summarised in the following two formal definitions.

DEFINITION. Let $X = \{1, 2, \ldots, n\}$ represent a set of social units, and let R_k stand for a relation of some type k (e.g., "is a friend of"), where $k = 1, 2, \ldots, p$. Let $(i, j) \in R_k$ indicate that unit i is R_k-related to unit j (e.g., "i names j as a friend"), where i and j are elements of X. R_k is a *binary relation* on the set X and may be formally described as a set of ordered pairs of elements of X. (General algebraic definitions may be found in Kurosh, 1963; definitions of some basic mathematical terms are also given in Appendix A.) The collection $\mathbf{R} = \{R_1, R_2, \ldots, R_p\}$ of relations on X is termed a *(multiple) network*, and the relations