The C_1 Contractions

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Let E be a Banach space and T a linear continuous operator on it. The operator is said to be a C_1*-contraction* if $\|T\| = 1$ and :

$$T^n x \not\to 0 , \quad n \to +\infty , \qquad \text{for all } x \neq 0.$$

The terminology "C_1 " is a shortening of Nagy-Foias teminology "$C_{1.}$ " ; see [7].

Here are three examples :

a) any isometry,

b) on the space $l_2(\mathbb{Z})$, equipped with the canonical basis $(e_n)_{n\in\mathbb{Z}}$, a weighted shift of the following type :

$$Te_n = w_n e_{n+1} ,$$

with $w_n = 1$ for ≥ 0, $w_n = 1/4$ for $n < 0$.

We refer to [2] for a detailed study of this operator.

c) Let K be a connected domain, with regular boundary, contained in the closed unit disk $\overline{D}$, and such that $\partial K \cap C$ is an interval I_1 (C is the unit circle). Let φ be a conformal map from D onto K°, and M_φ the operator of multiplication by φ on the Hardy space H^2. This operator is a C_1-contraction. Indeed, M_φ is of norm 1. Moreover, φ extends to an homeomorphism from $\overline{D}$ onto K ; let $I = \varphi^{-1}(I_1)$. Then, for every function f in H^2, which is not identically 0, we have :

$$\begin{aligned} \|M_\varphi^n f\|_2^2 &= \int_0^{2\pi} |\varphi^n(e^{i\theta})|^2 \, |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} \\ &\geq \int_I |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} , \end{aligned}$$

and this last quantity is strictly positive, because no function in H^2 (except 0) can vanish identically on a set of positive measure. We will come back later to this example, with more details.

As it is well-known, the usual bilateral shift : $Te_n = e_{n+1}$, $n \in \mathbb{Z}$, on $l_2(\mathbb{Z})$ has two types of Invariant Subspaces (see for instance Hoffman [6]) :

- type 1 : to every sequence $(a_j)_{j\in\mathbb{Z}}$, we associate the function

$$f(e^{i\theta}) = \sum_{j\in\mathbb{Z}} a_j e^{ij\theta} ,$$

which is in $L_2(\Pi, d\theta/2\pi)$. Let A be a measurable subset of Π, with $0 < P(A) < 1$ (P being the Haar measure on Π). Let :

$$F_A = \{(a_j) \in l_2(\mathbb{Z}) \ ; \ f = 0 \text{ on } A\}.$$

Then obviously, if (a_j) is in F_A, so is (Ta_j).

- type 2 : For $n \in \mathbb{Z}$, let

$$G_n = \{(a_j)_{j\in\mathbb{Z}} \ ; \ a_j = 0 \text{ if } j \leq n\}.$$

Obviously also, $TG_n \subset G_n$.

This second type has a more general description : let m be an inner function (see [6]), then $G = m \cdot H^2$ is the general form of such an Invariant Subspace.

More generally, we will say that a closed subspace $F \subset E$ is *invariant* for T if $TF \subset F$. The subspace F is said to be *non- trivial* if $F \neq 0$ and $F \neq E$. In the sequel, we omit the words "non-trivial" and speak about Invariant Subspaces.

The question whether every C_1-contraction has Invariant Subspaces is still open, even on Hilbert spaces. In what follows, we try to make a formal description of the two types of Invariant Subspaces of the usual shift, that is, to obtain a description which makes sense for a general C_1-contraction. This will be done in the next two paragraphs, under some specific hypotheses.

1. Invariant Subspaces of Functional Type.

Let $\mathcal{A}(\Pi)$ be the vector space of functions with absolutely convergent Fourier series :

$$\mathcal{A}(\Pi) = \{f = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \ ; \ \sum_{k\in\mathbb{Z}} |a_k| < \infty\} ,$$

which is an algebra under the norm :

$$\|f\|_{\mathcal{A}} = \sum_{k\in\mathbb{Z}} |a_k|$$

Let T be a C_1-contraction and f a function in $\mathcal{A}(\Pi)$. For any $m \in \mathbb{Z}$, we define :

$$\psi_m(f) = \sum_{k\geq -m} a_k T^{k+m} ,$$

and this series converges since $\|T\| = 1$.

We observe that the operators $(\psi_m(f))_{m\in\mathbb{Z}}$ are uniformly bounded ; indeed :

$$\|\psi_m(f)\| \leq \|f\|_{A} .$$

Therefore, the set :

$$F_f = \{x \in E \ ; \ \psi_m(f)x \to 0, \ m \to +\infty\},$$

is a closed subspace of E, which is invariant under T (this subspace is even hyper-invariant : that is, invariant also under all operators which commute with T).

Such a subspace is called "of functional type", because it arises from a function in $A(\Pi)$.

In the sequel, for convenience, we will assume T to be invertible (the condition upon the iterates $\|T^{-n}x_0\|$ may be replaced by a condition upon a chain of approximate backward iterates, when T is not invertible ; see [1]). Then we have :

Theorem 1 (B. B. [1]). – *If there exists a point x_0 such that*

$$\sum_{n\geq 0} \frac{\log \|T^{-n}x_0\|}{1+n^2} < +\infty \tag{1}$$

then T has non-trivial hyperinvariant subspaces of functional type. Moreover, the condition (1) is best possible for this type of invariant subspace : for any sequence $(\rho_n)_{n\geq 0}$, satisfying

$$\rho_n \geq 1 , \quad \rho_{m+n} \leq \rho_m \cdot \rho_n , \quad m,n \in \mathbb{N},$$

and

$$\sum_{n\geq 0} \frac{\log \rho_m}{1+m^2} < \infty ,$$

there exists an operator T with $\|T^{-m}\| = \rho_m$, such that, for every $x \neq 0$, there is $C(x) > 0$, with :

$$\|T^{-m}x\| \geq C(x)\rho_m ,$$

and this operator has no Invariant Subspace of functional type.

This result improves two previously known theorems :

- John Wermer (1954) made the assumption that :

$$\sum_{m\geq 0} \frac{\log \|T^{-m}\|}{1+m^2} < \infty ,$$

- Colojoara- Foiaş(1966) made the assumption that both T and ${}^tT$ are C_1 -contractions (this last assumption implies that some point x has a chain of *bounded* inverses, which is of course much stronger than (1)).

We refer the reader to [1] for the proof.

This theorem applies of course to the usual bilateral shift, and, when applied to it, it gives the Invariant Subspaces of the first type, which are called *spectral* subspaces. They correspond to the following basic idea : if f and g are two functions on Π, disjointly supported, such that $f(T)$ and $g(T)$ both make sense, the product $f(T) \circ g(T)$ will be zero, and if $f(T) \neq 0$, $g(T) \neq 0$, we have non-trivial Invariant Subspaces : $Ker\, f(T)$ is the required Invariant Subspace. Here, we cannot give a meaning to $f(T) = \sum_{k \in \mathbb{Z}} a_k T^k$, for any function f in $A(\Pi)$, because this series may diverge, but we replace this "ordinary" functional calculus by an asymptotic one, using $\psi_m(f)$.

So this Theorem provides a large supply of Invariant Subspaces ; however, it does not apply to all C_1-contractions. Condition (1) is required, and we have seen it was best possible. So we now turn to another approach, corresponding to the Invariant Subspaces of type 2.

2. The unitary extension of T.

(Most results in this paragraph are part of a join paper with Michel Rome [5].)

On the space E, we define a norm by the formula :

$$[\![x]\!] = \lim_{n \to +\infty} \|T^n x\|,$$

and denote by $\mathcal{E}$ the completion of E with respect to this norm. From the inequality :

$$[\![\cdot]\!] \leq \|\cdot\|,$$

we deduce that there is a continuous injection from E into $\mathcal{E}$. On the space $\mathcal{E}$, the operator T extends naturally to an isometry, which we denote by $\tilde{T}$. One sees easily that $\tilde{T}$ is surjective when T has dense range (which we may assume, if we are looking for Invariant Subspaces).

The space $\mathcal{E}$ is finitely representable in E. If E is a Hilbert space, so is $\mathcal{E}$, with the scalar product :

$$[x, y] = \lim_{m \to +\infty} \langle T^m x, T^m y \rangle ,$$

where $\langle .,. \rangle$ is the scalar product in E.

From now on, we assume in this paragraph that E is a Hilbert space. For x, y in E, we consider, for $k \in \mathbb{Z}$:

$$\lambda_k(x, y) = \lim_{m \to +\infty} \langle T^{m+k} x, T^m y \rangle ,$$

and the Fourier series

$$\sum_{k\in\mathbb{Z}} \lambda_{-k}(x,y)e^{ik\theta} . \tag{2}$$

Since the coefficients λ_k are bounded, this series defines a pseudo- measure. But this result can be improved :

We say that T is *completely non unitary* (in short c.n.u.) if there is no subspace F such that $TF = F$ and $T|_F$ is unitary. Then we have :

Proposition 2 (B. B. , M. Rome [5]). – *If T is c.n.u., the Fourier series (2) is that of an integrable function.*

Let's denote by $\Lambda_{x,y}$ this function. Since :

$$\lambda_{-k}(x,y) \;=\; [\tilde{T}^{-k}x,y] \;=\; \int_0^{2\pi} e^{-ik\theta}\Lambda_{x,y}(\theta)\,\frac{d\theta}{2\pi} ,$$

we obtain that, for every f, g in $A(\Pi)$:

$$[f(\tilde{T})x, g(\tilde{T})y] \;=\; \int_0^{2\pi} f(e^{i\theta})\bar{g}(e^{i\theta})\,\Lambda_{x,y}(\theta)\,\frac{d\theta}{2\pi} . \tag{3}$$

Therefore, the functions $\Lambda_{x,y}$ allow us to write scalar products of functions of T, at points x, y. For this reason, these functions will be called *representing functions* (for T) at points x, y.

If $y = x$, we just write Λ_x instead of $\Lambda_{x,x}$, and call it representing function at the point x.

This concept allows us to give an Invariant Subspace theorem, which generalizes those of type 2 :

Theorem 3 ([5]). – *If there is a point x such that :*

$$\int_0^{2\pi} \log\Lambda_x(\theta)\,\frac{d\theta}{2\pi} \;>\; -\infty \tag{4}$$

then :

$$dist(x, \overline{\text{span}}\{Tx, T^2x, \ldots\}) \;>\; 0. \tag{5}$$

Consequently, $\overline{\text{span}}\{Tx,T^2x,\ldots\}$ is the required non- trivial Invariant Subspace.

Again, this theorem applies to the usual bilateral shift (with $\Lambda_{e_0} = 1$), and gives the fact that $e_0 \notin \overline{\text{span}}\{e_1, e_2, \ldots\}$.

In order to study condition (4), we will list some properties of the functions Λ_x or $\Lambda_{x,y}$. We refer the reader to [5] for the proofs.

Assume that a point x_0 is cyclic for T, that is :

$$E = \overline{\text{span}}\{x_0, Tx_0, T^2x_0, \ldots\}.$$

Let $\mu_0 = \Lambda_{x_0} d\theta/2\pi$ be the corresponding measure. Then the support K_0 of μ_0 is exactly the spectrum of $\tilde{T}$. For a function $f \in L_\infty(K_0, \mu_0)$, the operator $f(\tilde{T})$ is well- defined, since $\tilde{T}$ is normal.

Proposition 4. – *Let x, y be in E, f, g in $L_\infty(K_0,\mu_0)$. Put $x' = f(\tilde{T})x$, $y' = g(\tilde{T})y$. Then :*

$$\Lambda_{x',y'} = f\bar{g}\,\Lambda_{x,y} .$$

Corollary. – *All the functions $\Lambda_{x,y}$ vanish a.e. on $C \setminus K_0$.*

This is clear : take $f = g = 1$ on K_0, and 0 on $C \setminus K_0$. Then $f(T) = I$, so $\Lambda_{x,y} = |f|^2\Lambda_x$, and this function is 0 on $C \setminus K_0$.

We observe also that $\sigma(\tilde{T}) \subset \sigma(T) \cap C$ (and this inclusion can be strict : see [3]). Therefore, if we want condition (4) to hold, we need $\sigma(\tilde{T}) = C$, and a fortiori, $\sigma(T) \supset C$.

Therefore, Theorem 3 applies only to the C_1-contractions such that $\sigma(T) \supset C$ (but even not to all of them).

Thus, so far, we have given two theorems describing formalizations of the Invariant Subspaces of the usual shift : type 1 and type 2 respectively. Each of them requires a specific assumption, so we may ask : does the combination of both cover all possible cases ? We will see that this is not the case. But before that, we mention several other applications of the functions Λ_x : they allow us to obtain a "reverse" functional calculus. For instance :

Proposition 5. –*Let*

$$\sum_{k\in\mathbb{Z}} a_k T^k x = z$$

be a convergent series in E, which means that :

$$\sum_{-M}^{N} a_k T^k x \to z, \qquad M, N \to +\infty.$$

Then the series $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$ is that of a function ϕ in $L_2(\Lambda_x\, d\theta/2\pi)$, in the sense that :

$$\sum_{-M}^{N} a_k e^{ik\theta} \to \phi, \qquad M, N \to +\infty,$$

in this space.

This statement can be wiewed as a reverse functional calculus, in the sense that, starting from a property of $f(T)$, we deduce a property of f, whereas in general one does the converse.

3. Study of an example.

We come back to the example we mentioned at the beginning : ϕ being a conformal map from D onto a regular domain $K°$, contained in D, such that $\partial K \cap C = I_1$ is an interval. We assume moreover that K does not contain the origin.

Therefore the operator M_ϕ of multiplication by ϕ in the space H^2 is invertible : ϕ is in H^∞ (even in $A(D)$), and is outer.

Moreover, for every function $f \in H^2$, there is a constant $C(f) > 0$ and an $\varepsilon > 0$ such that :

$$\|M_\phi^{-n}\|_2 \geq C(1+\varepsilon)^n , \qquad n \in \mathbb{N},$$

and therefore Theorem 1 does not apply to this operator. Moreover, one can see directly that it has no Invariant Subspaces of functional type.

It follows easily from Runge's theorem that, for every function f in H^2, not identically 0,

$$f \in \overline{\text{span}}\{\phi f, \phi^2 f, \ldots\}.$$

So property (5) does not hold. Indeed, also, since $\sigma(M_\phi) = K$, $\sigma(M_\phi) \cap C$ is not C, and the assumption of Theorem 3 does not hold.

Let $I = \phi^{-1}(I_1)$; this is the set where $|\phi| = 1$. Then one can see easily that $\mathcal{E} = L_2(I)$, and $\tilde{M}_\phi = M_\phi$, multiplication by ϕ on $L_2(I)$: this is a unitary operator.

The Invariant Subspaces for M_ϕ on $L_2(I)$ are easy to describe : they are of the form $L_2(A)$, where A is a measurable subset of I, and they are non-trivial if and only if $P(A) > 0$, $P(I \setminus A) > 0$. Another way of describing them is : the set of functions f which vanish a.e. on $I \setminus A$.

Since no function in H^2 (except 0) can vanish a.e. on such a set, we get :

Theorem 4 ([4]). – *If F is any Invariant Subspace of M_ϕ on $L_2(I)$, then $F \cap H^2$ is $\{0\}$ or H^2.*

So, no Invariant Subspace of M_ϕ comes from an Invariant Subspace of $\tilde{M}_\phi$. Since the extension $\tilde{T}$ has a meaning in Nagy-Foias dilation theory (it corresponds to the *-residual part of this extension, see [5] and [7]), we may say that no Invariant Subspace of this operator comes from Nagy-Foiaş dilation theory.

The Invariant Subspaces of M_ϕ are easy to describe : they are of the form $m \cdot H^2$, where m is an inner function. This is so because $M_\phi = \phi(M_{e^{i\theta}})$.

This leads us to the following comments : we now see that Theorems 1 and 3 are insufficient to describe all the C_1-contractions. We also observe that, though their conclusions are invariant if one replaces T by $f(T)$, their assumptions are not. Indeed, the usual shift satisfies the assumptions of both theorems, though its image by ϕ does not. It would be nice to have versions of these theorems which would be invariant under the operation $T \to f(T)$.

This is a short summary of a series of lectures given at the University of Illinois, Urbana-Champaign, during a special year in Modern Analysis, 1986-87. The author wishes to thank the Department of Mathematics for its nice hospitality, and Professor Earl Berkson for having arranged the invitation and the lectures.

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FACTORIZATION THEOREMS FOR INTEGRABLE FUNCTIONS

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Let $(Z,\mathscr{B},\mu)$ be a measure space and let $\mathscr{D}$ be a separable, complex Hilbert space. We denote by $L^2(\mu;\mathscr{D})$ the Hilbert space of all (classes of) measurable, square integrable functions $x : Z \longrightarrow \mathscr{D}$. For two functions $x,y \in L^2(\mu;\mathscr{D})$ we can define the function $x\cdot y \in L^1(\mu)$ by setting $(x\cdot y)(\zeta) = \langle x(\zeta),y(\zeta)\rangle$ for almost every $\zeta \in Z$. ($\langle\cdot,\cdot\rangle$ denotes the scalar product in any Hilbert space. For instance, for $x,y \in L^2(\mu;\mathscr{D})$ we have $\langle x,y\rangle = \int_Z \langle x(\zeta),y(\zeta)\rangle d\mu(\zeta)$.)

In this paper we study the possibility of solving, at least approximately, an equation of the form $x\cdot y = f$, where f is a given function in $L^1(\mu)$, and x,y are required to belong to a given subspace $\mathscr{H} \subset L^2(\mu;\mathscr{D})$. (See Theorem 10 for the precise statement.) Our results here strengthen and put in an abstract framework certain results of Brown [3] and the author [2]. These results were obtained in relation to operator theoretical problems. We will show in a subsequent paper how our results can be used to settle the conjecture made in [1] about the structure of contraction operators.

Throughout this paper $\mathscr{H}$ is a linear subspace of $L^2(\mu;\mathscr{D})$ satisfying the following condition.

1. ASSUMPTION. Given $\sigma \in \mathscr{B}$, $\mu(\sigma) > 0$, a positive number ϵ, and a finite number of vectors $\xi_1, \xi_2, \ldots, \xi_p \in L^2(\mu; \mathscr{D})$, there exists $z \in \mathscr{H}$, $z \neq 0$, such that

(i) z is essentially bounded, i.e., $z \in L^\infty(\mu; \mathscr{D})$;

(ii) $\|\chi_{Z\setminus\sigma} z\| < \epsilon \|\chi_\sigma z\|$; and

(iii) $\langle z, \xi_j \rangle = 0$, $j = 1, 2, \ldots, p$.

We need the following consequence of Assumption 1.

2. PROPOSITION. Let $f \in L^\infty(\mu)$ be a function such that $0 \le f \le 1$, let $\xi_1, \xi_2, \ldots, \xi_p \in L^2(\mu; \mathscr{D})$, and let $\epsilon > 0$. If $\|f\|_\infty > 1 - \epsilon$ then there exists $x \in \mathscr{H} \cap L^\infty(\mu; \mathscr{D})$ such that $\langle x, \xi_j \rangle = 0$, $1 \le j \le p$, and $\|(1-f)^{1/2} x\| < \frac{\epsilon}{1-\epsilon} \|f^{1/2} x\|^2$.

Proof. Choose $\epsilon' < \epsilon$ such that the set $\sigma = \{\zeta \in Z : f(\zeta) > 1 - \epsilon'\}$ has positive measure, and choose $\delta > 0$ such that

$$\frac{\epsilon' + \delta^2}{1-\epsilon'} < \frac{\epsilon}{1-\epsilon}.$$

By Assumption 1 we can find $x \in \mathscr{H} \cap L^\infty(\mu; \mathscr{D})$, $x \neq 0$, with $\langle x, \xi_j \rangle = 0$, $1 \le j \le p$, such that $\|\chi_{Z\setminus\sigma} x\|^2 < \delta \|\chi_\sigma x\|^2$. We will show that this x satisfies the conclusion of our proposition. We have

$$\|f^{1/2} x\|^2 \ge \int_\sigma f(\zeta) \|x(\zeta)\|^2 d\mu(\zeta) \ge (1-\epsilon') \|\chi_\sigma x\|^2,$$

and hence

$$\|(1-f)^{1/2} x\|^2 = \int_\sigma (1 - f(\zeta)) \|x(\zeta)\|^2 d\mu(\zeta) + \int_{Z\setminus\sigma} (1 - f(\zeta)) \|x(\zeta)\|^2 d\mu(\zeta)$$