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Membership of Hankel operators on planar domains in unitary ideals

by

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§1. *Introduction*

A *Hankel matrix* is a matrix of the form

$$A = (c_{n+k})_{n,k \geq 0}$$

i.e., $a_{n,k} = c_{n+k}$ for $n,k \geq 0$. These matrices are naturally related to analytic functions f on the unit disk Δ via

$$f(z) = \sum_{n=0}^{\infty} \bar{c}_n z^n$$

The most efficient way of studying the properties of the Hankel matrix A as an operator on ℓ_2 is via the action of the *Hankel operator*

$$H_f = (I-P)M_{\bar{f}}P$$

on $L^2(\mathbb{T})$, where $\mathbb{T} = \partial\Delta = \{z\in\mathbb{C};\ |z| = 1\}$, P is the orthogonal projection onto the Hardy space H^2, and $M_{\bar{f}}$ is the operator of multiplication by $\bar{f}$. The connection between the two objects is that the matrix of H_f in the bases $\{z^n\}_{n\geq 0}$ in H^2 and $\{\bar{z}^k\}_{k\geq 1}$ in $L^2 \ominus H^2 = (H^2)^{\perp}$ is the Hankel matrix A.

Similar operators, also called Hankel operators, are studied in a wider context (General planar domains, non–analytic symbols, higher dimensions etc.). We discuss here the extension of the theory to planar domains.

Let Ω be a domain in the complex plane and μ a finite positive measure on Ω: Let $A^2(\mu) = A^2(\Omega,\mu)$ denote the space of all analytic functions on Ω which belong to $L^2(\mu) = L^2(\Omega,\mu)$. We assume that convergence in $A^2(\mu)$ implies uniform convergence on compact subsets of Ω. Thus $A^2(\mu)$ is closed in $L^2(\mu)$ and point evaluations are continuous lienar functionals on $A(\mu)$.

Let $P: L^2(\mu) \to A^2(\mu)$ be the orthogonal projection. For an analytic function f on Ω we consider the *Hankel operator*

$$H_f = (I-P)M_{\bar{f}}P$$

We sometimes identify H_f with its restriction to $A^2(\mu)$.

The *main theme* in studying Hankel operators is *the connection between the size of the operator* H_f — (boundness, compactness, membership in Schatten classes, S_p, etc.) and the *size of the symbol* f (rate of increase f(z) as $z \to \partial\Omega$, degree of smoothness of boundary values, rate of approximation by "nice" functions, etc.).

We would like to report here on some recent works on membership of Hankel operators in unitary ideals in the context of the unit disk [AFP2] and planar domains of finite connectivity [AFP3].

These notes are an expanded version of the talks given by the author at the University of Illinois at Urbana–Champaign to the participants of the Special Year in Modern Analysis. The author thanks Professor J. Bourgain and E. Berkson for arranging his visit.

§2. *Hankel operators in $L^2(\mathbb{T})$, Schatten p–classes, BMOA, VMOA and Besov–p spaces*

The conditions for boundedness and compactness of the Hankel operators H_f on $L^2(\mathbb{T})$, f analytic on Δ, were found by Z. Nehari and P. Hartman respectively. To formulate these results in our terminology, let

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \; ; \; a,z \in \Delta$$

φ_a is the *Mobius function* which preserves Δ and interchange 0 and a. An analytic function f on Δ is in the space *BMOA* (analytic functions with bounded mean oscillation) if

$$\|f\|_{BMOA} := \sup_{a\in\Delta} \|f\circ\varphi_a - f(a)\|_{H^2} < \infty$$

f belongs to *VMOA* (analytic functions of vanishing mean oscillation) if f$\in$BMOA and $\lim_{a\to\partial\Delta} \|f\circ\varphi_a - f(a)\|_{H^2} = 0$. Thus we have adopted the so–called Garsia norm to define BMOA and VMOA.

Theorem 2.1 [N] *The Hankel operator* H_f *on* $L^2(\mathbb{T})$ *is bounded if and only if* f *is in* BMOA.

Theorem 2.2 [H] *The Hankel operator* H_f *on* $L^2(\mathbb{T})$ *is compact if and only if* f *is in* VMOA.

The space of compact operators from a Hilbert space M into a Hilbert space N is denoted by $S_\infty(M,N)$, or simply S_∞ in case M and N are understood. The singular numbers of $T \in S_\infty$ are the eigenvalues of $(T^*T)^{\frac{1}{2}}$, $s_n(T) = \lambda_n((T^*T)^{\frac{1}{2}})$, arranged in a non-increasing ordering, counting multiplicity. The Schatten p-classes $S_p = S_p(M,N)$, $0 < p < \infty$, consist of all $T \in S_\infty$ for which

$$\|T\|_{S_p} := [\text{trace } (T^*T)^{p/2}]^{1/p} = (\sum_{n=1}^{\infty} s_n(T)^p)^{1/p}$$

is finite. More generally, if E is a symmetric sequence space, the associated unitary ideal $S_E = S_E(M,N)$ consists of those $T \in S_\infty$ for which $(s_n(t))_{n=1}^{\infty} \in E$, normed by

$$\|T\|_{S_E} = \|(s_n(T))\|_E$$

Clearly, $T \in S_E(M,N)$ if and only if $UTV \in S_E\ (M_1, N_1)$ for all unitary operators V from M_1 onto M and U from N onto N_1, and $\|T\|_{S_E} = \|UTV\|_{S_E}$. This explains the name "unitary ideals". The space S_2 (= Hilbert Schmidt operators) is a Hilbert space with respect to the inner product

$$(A,B)_{S_2} = \text{trace } (AB^*)\ .$$

This pairing is clearly unitarily invariant. With respect to this pairing, $S_\infty(M,N)^* = S_1(M,N)$, $S_1(M,N)^* = B(M,N)$ and

$$S_p(M,N)^* = S_q(M,N)\ , 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1\ .$$

See [GK] and [S] for more information on S_p and unitary ideals in general.

An analytic function f on Δ belongs to the *Besov space* B_p $(= B^{1/p}_{p,p})$, $1 < p \le \infty$, if

$$\|f\|_{B_p} := (\int_\Delta |f'(z)|^p (1-|z|^2)^{p-2} dA(z))^{1/p}$$

is finite. Here dA is the normalized Lebesgue's measure of Δ. Thus $f \in B_p$ if and only if the Möbius–invariant derivative $(1-|z|^2)f'(z)$ is in $L^p(\mu)$, where $d\mu(z) = dA(z)/(1-|z|^2)^2$ is the Möbius–invariant measure on Δ. The space B_∞ is known also as the *Bloch space* and B_2 is the *Dirichlet space*.

For $0 < p \leq 1$ the space B_p consists of all analytic functions f on Δ for which

$$\int_\Delta (1-|z|^2)^{pn-2} \; |f^{(n)}(z)|^p dA(z) \quad < \quad \infty$$

where $1/p < n$ is an integer (It is well known that the definition is independent of the choice of n).

For $1 < p \leq \infty$, B_p are Banach spaces modulo constant functions. B_2 is Hilbert space, with a Möbins–invariant inner–product

$$(f;g)_{B_2} = \int_\Delta f'(z)\overline{g'(z)} \; dA(z) \; .$$

With respect to this pairing one has, up to an equivalent norm,

$$B_p^* = B_q; \qquad\qquad 1 < p < \infty, \;\; \frac{1}{p} + \frac{1}{q} = 1.$$

The space B_1 admits the equivalent description [AFP1] as those analytic functions f on Δ which admit a representation

$$f = \sum_j \lambda_j \varphi_{a_j}; \quad a_j \in \Delta \; , \quad \sum_j |\lambda_j| < \infty \; .$$

With respect to the norm

$$\|f\|_{B_1} = \inf \left\{ \sum_j |\lambda_j|; \; f = \sum_j \lambda_j \varphi_{a_j}, \quad \lambda_j \in \mathbb{C}, \quad a_j \in \Delta \right\}$$

and the B_2–inner product we have [AFP1]

$$B_1^* = B_\infty \; , \; b_\infty^* = B_1$$

where

$$b_\infty := \{f \in B_\infty; \; \lim_{|z| \to 1} (1-|z|^2)(f'(z) = 0\}$$

is the so–called *little Bloch space*.

See [P2] and [BL] for more information on Besov spaces.

Theorem 2.3 *Let* $0 < p < \infty$ *and let* f *be analytic on* Δ. *Then the Hankel operator* H_f *on* $L^2(\mathbb{T})$ *belongs to* S_p *if and only if* $f \in B_p$.

This result is due mainly to V.V. Peller. In [Pel1] the theorem is proved for $1 \leq p < \infty$. A little later, but independently, the result was proved for p=1 in [CR] and for $1 < p < \infty$ in [R1]. Peller [Pel4] and Semmes [Se] independently proved the result for $0 < p < 1$.

Let us mention also an older result of Kronecker.

Theorem 2.4: *The Hankel operator* H_f *on* $L^2(\mathbb{T})$ *is of finite rank* n *if and only if* f *is a rational function of degree* n, *with poles outside* $\overline{\Delta}$.

The theory of Hankel operators on $L^2(\mathbb{T})$ is very rich and has many applications. See [Po], [P1] and [Ni] for general surveys and extensive literature. [Pel1], [Pel4] and [PK] contain many applications of Hankel operators to approximation theory and Gaussian processes. Peller studied some other problems related to Hankel operators on $L^2(\mathbb{T})$ and unitary ideals. In [Pel4] he characterized the Hankel operators in the Lorentz ideal S_{pq}, and in [Pel2] and [Pel3] he studied the continuity properties of the averaging projection on the set of Hankel matrices.

§3. Hankel operators on weighted Bergman spaces on the unit disk

For $\alpha > -1$ let us consider the normalized weighted area measure on the unit disk Δ

$$d\sigma_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$$

Let $L^{2,\alpha} = L^2(\sigma_\alpha)$ and let $A^{2,\alpha} = A^2(\sigma_\alpha)$ be the subspace of analytic functions in $L^{2,\alpha}$. The spaces $A^{2,\alpha}$ are known as *weighted Bergman spaces* (for $\alpha=0$ we obtain the usual Bergman space). The reproducing kernel (*Bergman kernel*) of $A^{2,\alpha}$ is

$$K^{\alpha}(z,w) = K^{\alpha}_{w}(z) = (1-z\bar{w})^{-(\alpha+2)}; \quad z,w \in \Delta.$$

The orthogonal projection (*Bergman projection*)

$$P_{\alpha}: L^{2,\alpha} \longrightarrow A^{2,\alpha}$$

is given for $g \in L^{2,\alpha}$ by

$$(P_{\alpha}g)(z) = (g, K^{\alpha}_{z})$$

Since $\alpha > -1$ is fixed we shall write K and P instead of K^{α} and P_{α}.

If f is an analytic function on Δ, the *Hankel operator with symbol* f *on* $L^{2,\alpha}$ is

$$H_f = (I-P)M_{\bar{f}}P$$

(with the usual convention of considering H_f as an operator from $D(H_f) \subseteq A^{2,\alpha}$ into $(A^{2,\alpha})^{\perp}$).

The main results of [AFP2] concerning the relationship between the sizes of H_f and that of f, are the following.

<u>*Theorem 3.1*</u> (*a*) H_f *is bounded if and only if* $f \in B_{\infty}$;
(*b*) H_f *is compact if and only if* $f \in b_{\infty}$.

<u>*Theorem 3.2*</u> (*a*) *For* $1 < p < \infty$, $H_f \in S_p$ *if and only if* $f \in B_p$;
(*b*) $\|H_f\|_{S_2} = \|f\|_{B_2}$

The *Macaev sequence space* M is the Banach space of all sequences $x = (x_n)_{n=1}^{\infty}$ for which

$$\|x\|_M := \sup_N \frac{\sum_{n=1}^{N} x_n^*}{\sum_{n=1}^{N} \frac{1}{n}}$$

is finite. Here $\{x_n^*\}_{n=1}^{\infty}$ is the non-increasing rearrangement of $\{|x_n|\}_{n=1}^{\infty}$. The corresponding unitary ideal S_M is called the *Macaev ideal.* Clearly $S_1 \subsetneq S_M \subsetneq S_p$ for every $p > 1$.

Theorem 3.3: *Let* $f \in B_1$, *then* $H_f \in S_M$

Theorem 3.4: S_M *is the minimal normed unitary ideal containing a non-zero Hankel operator* H_f, f *analytic.*

The case $\alpha=0$ in Theorem 3.1 was proved earlier by S. Axler [A]. Comparing Theorem 3.1 with Theorems 2.1 and 2.2, we see that in the context of the weighted Bergman spaces (i.e. the weighted *area* measure versus the Lebesgue's measure on the *circle*) *the Bloch space* B_∞ replaces BMOA and the little Bloch space b_∞ replaces VMOA. In fact it is not hard to see that

$$\|f\|_{B_\infty} \approx \sup_{a \in \Delta} \|f \circ \varphi_a - f(a)\|_{L_{2,\alpha}}$$

i.e. the usual Bloch norm is equivalent to the "Garsia–BMOA norm" with respect to the weighted area measure σ_α.

Invariance

We begin our survey of the proof of Theorems 3.1–3.4 with a few words on invariance. The *Mobius group* $G = \text{Aut}(\Delta)$ consists of all biholomorphic automorphisms of Δ. These are known to have the form

$$\psi = \lambda \varphi_a \text{ , } a \in \Delta, \ \lambda \in = \partial\Delta,$$

where

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \text{ ; } z \in \Delta$$

A Banach space X of analytic functions on Δ is *Mobius invariant* if G operates on X by compositions as a strongly–continuous group of isometries. It is not hard to see that the spaces B_p $(1 < p < \infty)$ and VMOA are Möbius–invariant, and that B_∞ and BMOA are Möbius–invariant if the continuity of the group action is understood in the w^*–topology.

Next, we define an action of the Möbius group on $L^{2,\alpha}$ via

$$V_\psi(f) = (\psi')^{1+\alpha/2}\cdot(f\circ\psi) \quad ; \qquad \psi\in G,\ f\in L^{2,\alpha}$$

We have

<u>*Proposition 3.5*</u> *(a) For* $\psi\in G$, V_ψ *is a unitary operator on* $L^{2,\alpha}$;

(b) $V_{\varphi\psi} = V_\varphi V_\psi$; $\quad \varphi, \psi\in G$;

(c) $V_{\psi_0} = I \qquad (\psi_0(z) \equiv z)$;

(d) $V_\psi P = PV_\psi$; $\quad \psi\in G$,

so V_φ *restricts to a unitary operator of* $A^{2,\alpha}$ *and of* $(A^{2,\alpha})^\perp$.

<u>*Proposition 3.6*</u> *For* $\varphi\in G$ *and an analytic function* f *on* Δ *holds*

$$H_{f\circ\varphi} = V_\varphi H_f V_\varphi^{-1}$$

In particular $H_{f\circ\varphi}$ *is unitarily equivalent to* H_f.

Using these elementary propositions one obtains the following important property of the map $\mathscr{H}$ which sends an analytic function f on Δ to the corresponding Hankel operator H_f. Here a "minimal" unitary ideal is a one in which the finite rank operators are dense, and a "maximal" unitary ideal is the dual of a minimal one.

<u>*Corollary 3.7*</u>: *Let* S *be a minimal unitary ideal of operator on* $L^{2,\alpha}$. *Then* $\mathscr{H}^{-1}(S)$: $=$ {f *analytic in* Δ; $H_f\in S$} *is a Mobius–invariant space. If* S *is maximal, then* $\mathscr{H}^{-1}(S)$ *is Mobius–invariant with respect to its* w^**–topology.*

<u>*The proof of Theorem 3.1*</u>

One begins with the observation that for an analytic function f on Δ and an analytic function g in the domain of H_f,

$$(H_f g)(z) = \int_\Delta (\overline{f(z)} - \overline{f(w)})K(z,w)g(w)d\sigma_\alpha(w)$$

Thus H_f is closely related to the integral operator T_{Q_f} on $L^{2,\alpha}$ with kernel

$$Q_f(z,w) := (\overline{f(z)} - \overline{f(w)})K(z,w)$$

If $f \epsilon B_\infty$ then for all $z,w \epsilon \Delta$

$$|f(z) - f(w)| \le \|f\|_{B_\infty} \; d(z,w),$$

where

$$d(z,w) = \frac{1}{2} \log \left(\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} \right)$$

is the hyperbolic distance between z and w. With $u(z) = (1-|z|^2)^{-(\frac{\alpha+1}{2})}$ we see that for all $z \epsilon \Delta$

$$\int_\Delta |Q_f(z,w)| u(w) d\sigma_\alpha(w) \le C \; u(z)$$

Schur's lemma (see [HS] or [AFP2]) now implies that T_{Q_f} is bounded; hence H_f is bounded as well.

To show that boundedness of H_f implies $f \epsilon B_\infty$ we introduce a family of seminorms

$$|\!|\!| f |\!|\!|_a = \|H_f k_a\|_{L^{2,\alpha}} \;, \; a \epsilon \Delta \;,$$

where

$$k_a = K_a / \|K_a\|_{L^{2,\alpha}}$$

These seminorms are Möbius–invariant.

$$|\!|\!| f \circ \varphi |\!|\!|_a = |\!|\!| f |\!|\!|_{\varphi(a)} \quad ; \; a \epsilon \Delta \;, \; \varphi \epsilon G$$

and clearly

$$|\!|\!| f |\!|\!|_0 = \|f - f(0)\|_{L^{2,\alpha}} \ge C|f'(0)|.$$

Thus