Advanced mathematical methods for engineering and science students
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Preface

This book contains a selection of advanced topics suitable for final year undergraduates in science and engineering, and is based on courses of lectures given by one of us (G. S.) to various groups of third year engineering and science students at Imperial College over the past 15 years. It is assumed that the student has a good understanding of basic ancillary mathematics. The emphasis in the text is principally on the analytical understanding of the topics which is a vital prerequisite to any subsequent numerical and computational work. In no sense does the book pretend to be a comprehensive or highly rigorous account, but rather attempts to provide an accessible working knowledge of some of the current important analytical tools required in modern physics and engineering. The text may also provide a useful revision and reference guide for postgraduates.

Each chapter concludes with a selection of problems to be worked, some of which have been taken from Imperial College examination papers over the last ten years. Answers are given at the end of the book.

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Notes to the reader

1. The symbol \( \ln \) denotes logarithm to base \( e \).
2. The end of a worked example is denoted by \( \blacktriangle \).