Chaotic evolution and strange attractors

The statistical analysis of time series for deterministic nonlinear systems

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Descriptions of turbulence

An important question arising from many experimental situations (for example, studying turbulent behavior in a fluid flow) is the following: How does one explain a situation in which one gets a signal (i.e. a time series) which is nonperiodic, indeed a chaotic signal?

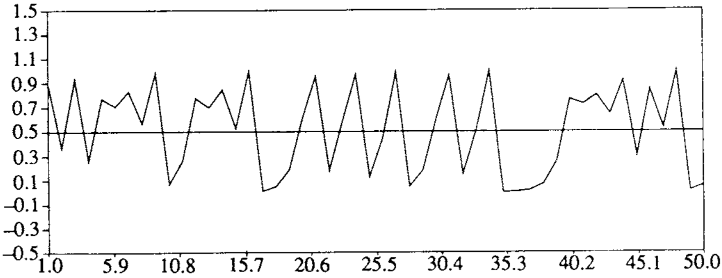
One idea is the following: if one has a system which gives a 'noisy' signal, that is nonperiodic and irregular, this means that there must be some inputs which are also noisy and nonperiodic. The formalisation of this idea lies on the so-called *stochastic evolution equations*, namely equations of the form

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = F(\mathbf{x}(t)) + w(t), \tag{2}$$

where $w(t)$ is the noise term (e.g. a stochastic process).

As far as hydrodynamical systems are concerned, the (infinite) set of observables $\mathbf{x}(t)$ will represent the modes amplitude (to be defined later), and an experiment carried out with a sufficiently excited system (for example high values of the Rayleigh number) yields a situation just like the one sketched in Fig. 1. Let us stress that turbulence is the

Fig. 1. A chaotic signal obtained by the simple deterministic difference equation $x_{n+1} = 4x(1 - x_n)$. Any correlation test would reveal rapid decorrelation between successive iterations, making this sequence akin to a random sequence.



type of physical phenomena where one gets such a noisy signal. It is interesting to note that some people have considered that the theory of turbulence must necessarily be of the form expressed by (2).

Nevertheless, there are other kinds of explanations of a nonperiodic signal. One of them involves the presence of many oscillators (Landau). According to this theory, the time evolution of the physical parameter describing a turbulent fluid is given (asymptotically) by:

$$\mathbf{x}(t)=\mathbf{f}(\omega_1 t,\omega_2 t,\ldots,\omega_k t)=\mathbf{f}(\phi_1,\phi_2,\ldots,\phi_k), \tag{3}$$

where $\mathbf{f}$ is a periodic function of period 2π in each of its arguments, and $\omega_1,\omega_2,\ldots,\omega_k$ are rationally independent frequencies; $\mathbf{x}(t)$ is then a *quasiperiodic* function of t. The motion (3) describes a k-dimensional torus T^k (i.e. the product of k circles) embedded in R^m and constitutes what is called a *quasiperiodic attractor.* Such attractors are a generalisation of periodic orbits, but they describe motions which look indeed nonperiodic and very irregular, thus suggesting turbulence.

In general, in a quasiperiodic time evolution it does not make sense to specify which are *the* frequencies $\omega_1,\ldots,\omega_k$ of the motion. In fact, if we look at the Fourier transform of the signal $\mathbf{x}(t)$:

$$\mathbf{x}(t)=\sum_{n_1\ldots n_k}\hat{x}_{n_1\ldots n_k}\exp[i(\omega_1 n_1+\cdots+\omega_k n_k)t] \tag{4}$$

we can see that all the harmonics are present. Therefore, we can choose any other set of 'basic' frequencies of the form:

$$\omega_j'=n_{j1}\omega_1+\cdots+n_{jk}\omega_k \text{ with } j=1\ldots k, \tag{5}$$

where the matrix (n_{ij}) has integer entries and determinant ± 1. Then, the *number k* of frequencies of a quasiperiodic motion is defined as the minimum number of rationally independent frequencies of the form (5) which are present in the Fourier transform (4). This is just what is referred to as the number of *modes* of the system, and, in a sense, it plays the role of effective 'dimension' of a quasiperiodic motion.

We shall see later that, as far as nonlinear dynamical systems are concerned, even a finite-dimensional motion need not be quasiperiodic (indeed it may be chaotic), and the concept of 'number of modes' must be replaced by other concepts such as 'information dimension', or 'number of non-negative characteristic exponents'.

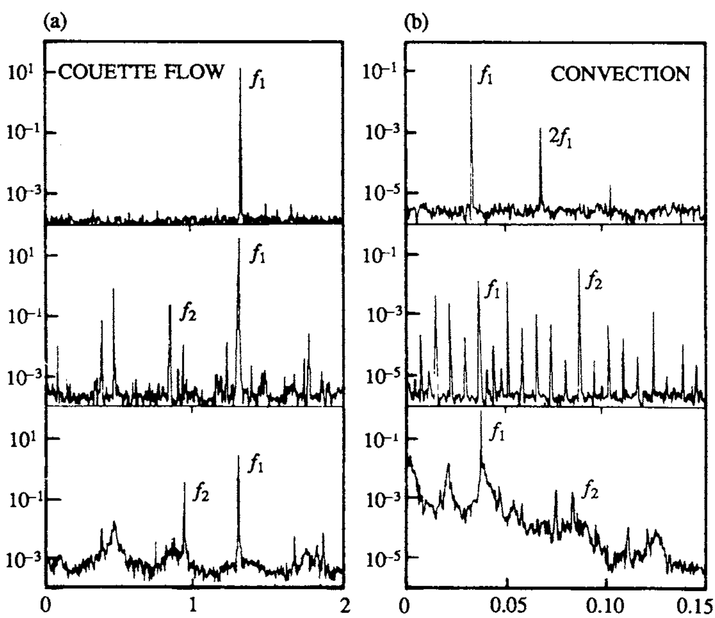
Now, starting from the evolution $\mathbf{x}(t)$ of coordinates, we can introduce a more general *observable*:

$$G(t) = g(\mathbf{x}(t)), \tag{6}$$

where g: $R^m \to R$ is a differentiable function. Then, a first indicator of the qualitative nature of the motion is the *power spectrum* or *frequency spectrum* (see Fig. 2), which measures the amount of energy per unit time contained in the signal $g(t)$ as a function of the frequency ω:

$$S(\omega) = (\text{const.}) \lim_{T\to\infty} \frac{1}{T} \left| \int_0^T G(t)\exp(-\mathrm{i}\omega t)\mathrm{d}t \right|^2. \tag{7}$$

Fig. 2. (a) The spectra of the velocity of a fluid occupying the interval between two coaxial cylinders, when the inner cylinder is rotated at three different speeds (Couette flow). (b) The spectra of the convective heat transport in a liquid layer heated from below at three different heating intensities (Rayleigh–Bénard convection). In both cases, from the top to the bottom we see a periodic spectrum, a quasiperiodic spectrum and a continuous spectrum. From Gollub and Swinney (1978).



In the particular case of quasiperiodic time evolutions we find that $S(\omega)$ is formed of discrete peaks corresponding to the basic frequencies $\omega_1 \ldots \omega_k$ and their linear combination with integer coefficients

$$S(\omega)=\sum_{n_1 \ldots n_k} c_{n_1 \ldots n_k}\delta\left(\omega-\sum_{i=1}^{k} n_i\omega_i\right). \tag{8}$$

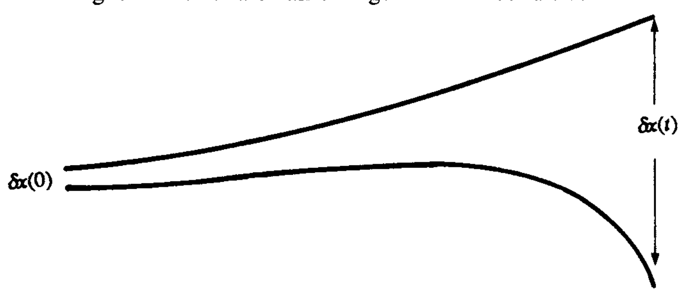
However, in practice one never computes the limit in (7), therefore the peaks have, at least, a width $2\pi/T$. Moreover, the structure of the spectrum, namely the number of peaks which are really visible, changes with the choice of the function g, and experimentally it is difficult to observe more than a few independent frequencies.

Now, the main problem with a quasiperiodic theory of turbulence (putting several oscillators together) is the following: when there is a nonlinear coupling between the oscillators, it very often happens that the time evolution does not remain quasiperiodic. As a matter of fact, in this latter situation, one can observe the appearance of a feature which makes the motion completely different from a quasiperiodic one. This feature is called *sensitive dependence on initial conditions* and turns out to be the conceptual key to reformulating the problem of turbulence.

Let us assume that the system has a deterministic time evolution defined by an autonomous ordinary differential equation like (1). Let $\mathbf{x}(t)$ be the solution of such an equation corresponding to the initial condition $\mathbf{x}(0)$. If we change slightly the position of the initial point: $\mathbf{x}(0)\rightarrow\mathbf{x}'(0)=\mathbf{x}(0)+\delta\mathbf{x}(0)$, the point at time t will also be changed (see Fig. 3).

Generally speaking, from the continuity of the solutions of an

Fig. 3. Effect of a small change of initial condition.



ordinary differential equation with respect to the initial conditions, one expects that, if $\delta\mathbf{x}(0)$ is small, $\delta\mathbf{x}(t)$ is also small. But, what may happen is that, when the time becomes large, the small initial distance grows anyway, and it may grow exponentially fast: $\delta\mathbf{x}(t) \sim \delta\mathbf{x}(0)\exp(\lambda t)$, where λ measures the mean rate of divergence of the orbits. In this case the motion, although purely deterministic, has those stochastic features referred to as *chaos*. In fact, in all those cases in which the initial state is given with limited precision (if we assume that the space-time is continuous this is always the case because a generic point turns out to be completely specified only by an infinite amount of information, for example by an infinite string of numbers), we can observe a situation in which, when time becomes large, two trajectories emerge from the 'same' initial point. So, even though there is a deterministic situation from a mathematical point of view (the uniqueness theorem for ordinary differential equations is not in question), nevertheless the exponential growth of errors makes the time evolution self-independent from its past history and then nondeterministic in any practical sense.

However, it is quite obvious that a quasiperiodic motion represented by a solution like (3) cannot exhibit sensitive dependence on initial conditions. A small change in initial conditions simply replaces the arguments $\omega_1 t, \ldots, \omega_k t$ by $\omega_1 t + \alpha_1, \ldots, \omega_k t + \alpha_k$, where $\alpha_1, \ldots, \alpha_k$ are small.

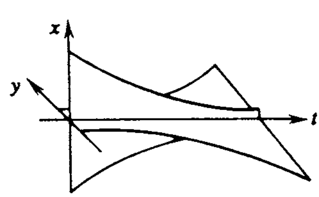
We shall see that a theoretical approach which wants to describe in a coherent fashion some hydrodynamic phenomena, in particular which wants to explain a noisy signal like the one in Fig. 1, has to put itself in the picture of deterministic noise; so that the meaning of turbulence will get close to those of chaos, dynamical instability and strange attractors.

The existence of sensitive dependence on initial conditions was first noticed by Hadamard, at the end of the last century, when studying the geodesic flow on compact surfaces of constant negative curvature. Such a compact surface M is obtained from the Lobachevsky plane by making certain identifications. The Lobachevsky plane itself may be viewed as the complex upper half-plane $\mathrm{Im}z > 0$ with the metric $ds^2 = (dx^2 + dy^2)/y^2$, where $z = x + iy$. The geodesics are then the half-circles and straight lines orthogonal to the x-axis. Returning to the compact manifold M, we let T_1M be the set of vectors tangent to M and of length 1. An element (x,u) of T_1M is thus a vector tangent to M at some point x. There is a unique oriented geodesic of M passing through x and tangent to u. Let x_t be the point at distance t of x on the geodesic, and u_t the unit vector tangent to the geodesic at x_t. The map which sends (x,u) to (x_t,u_t) is a diffeomorphism f^t of M_1T (differentiable map with differentiable inverse), and the family (f^t) is the geodesic flow. This is a particular case of an Anosov flow on T_1M, which means that it has the following remarkable property: the tangent spaces to T_1M can be written as direct sums $E^s + E^u + E^0$ where E^0 is one-dimensional in the direction of the flow and E^u (respectively E^s) is exponentially expanded (respectively contracted) by the flow. We shall come back later to a more precise discussion of these properties but, for the moment, we can get an intuitive idea of the dynamical instability of a hyperbolic flow by looking at Fig. 4.

After Hadamard had realised the possible presence of dynamical instability due to sensitive dependence on initial conditions, Poincaré and Duhem wrote popular texts explaining the philosophical importance of this feature. Then, although this was not forgotten in mathematics, it seems that for a long period it was forgotten by physicists.

The rediscovery of sensitive dependence on initial conditions, about 25 years ago, corresponded to the availability of electronic computers, which allowed the 'step-by-step' computation of the solutions of differential equations. This kind of computation has shown, and continues to show with strong evidence, that many time

Fig. 4. The t-axis (the direction of the flow) is the intersection of two surfaces of trajectories approaching it as $t \to \infty$ (the (x,t) surface) and as $t \to -\infty$ (the (y,t) surface); the remaining trajectories move away both for $t \to \infty$ and $t \to -\infty$. From Arnold (1980).



evolutions related to physical as well as technological problems do exhibit chaotic behavior, i.e. sensitive dependence on initial conditions.

A celebrated example is the Lorenz system (see Lorenz, 1963), a nonlinear time evolution in R^3 defined by the equations:

$$\begin{aligned}\frac{dx}{dt} &= -\sigma x + \sigma y \\ \frac{dy}{dt} &= -xy + rx - y \\ \frac{dz}{dt} &= xy - bz.\end{aligned} \tag{9}$$

These equations are obtained by truncation of the Navier–Stokes equation, and give an approximate description of a horizontal fluid layer heated from below. The warmer fluid formed at the bottom tends to rise creating convection currents. This is similar to what happens in the earth's atmosphere. For sufficiently intense heating the time evolution has sensitive dependence on initial conditions, thus representing a very irregular and chaotic convection. This fact was used by Lorenz to justify the so-called 'butterfly effect', a metaphor of the imprecision of weather forecasting. Actually, how this system really relates to turbulence (in particular to atmospheric turbulence) is not known yet, but what is remarkable is that it gives rise to a type of attractor which is nonclassical (neither periodic nor quasiperiodic).

The geometrical object described by the points of the trajectory is our first example of a *strange attractor*. We shall worry later about the mathematical definition of a strange attractor; for the moment, let us say that it is an infinite set of points (of which Fig. 5 shows a subset), in an m-dimensional space (here $m = 3$), which represents the asymptotic behavior of a chaotic system.

It is worth remarking that a time evolution which is chaotic in the sense sketched above usually exhibits a continuous power spectrum. On the other hand, the power spectrum of the velocity of a turbulent fluid is found experimentally to be continuous (see, for example, the bottom of Fig. 2). At first, this fact was attributed to the accumulation of a large number of independent frequencies, but accurate experiments by Swinney have shown that, when the fluid is excited above a certain threshold, a sharp transition towards a really continuous spectrum takes place.

So far, the power spectrum is the first indicator we have introduced which enables us to *distinguish* regular and chaotic motions. However, it is not really a 'good' indicator for the specific analysis of chaotic motions on strange attractors because the 'dimension' of a chaotic motion is no longer related to the number of independent frequencies (i.e. the number of modes); rather, it constitutes an important statistical feature in itself, which can be related both to the temporal aspect of chaos (number of positive characteristic exponents) and to its geometrical aspect (scale laws characterising the self-similar structure of strange attractors).

Fig. 5. The trajectory of the points (x,y,z) corresponding to the solutions of (9) with initial conditions near (0,0,0) and with $\sigma = 10$, $b = 8/3$, $r = 28$. From Lanford (1977).