ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Navier–Stokes Equations and Turbulence

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2001

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13 pt. *System* AMS-TEX [FH]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Navier–Stokes equations and turbulence / C. Foias ... [et al.].
p. cm. – (Encyclopedia of mathematics and its applications; v. 83)
Includes bibliographical references and index.
ISBN 0-521-36032-3
1. Turbulence. 2. Navier–Stokes equations. I. Foias, Ciprian.
II. Series.

QA913.N38 2001
532′.0527 – dc21 00-053008

ISBN 0 521 36032 3 hardback

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Introduction and Overview of Turbulence

Introduction

In this chapter we first briefly recall, in Section 1, the derivation of the Navier–Stokes equations (NSE) starting from the basic conservation principles in mechanics: conservation of mass and momentum. Section 2 contains some general remarks on turbulence, and it alludes to some developments not presented in the book. For the benefit of the mathematically oriented reader (and perhaps others), Section 3 provides a fairly detailed account of the Kolmogorov theory of turbulence, which underlies many parts of Chapters III–V. For the physics-oriented reader, Section 4 gives an intuitive introduction to the mathematical perspective and the necessary tools. A more rigorous presentation appears in the first half of Chapter II and thereafter as needed. For each of the aspects that we develop, the present chapter should prove more useful for the nonspecialist than for the specialist.

1 Viscous Fluids. The Navier–Stokes Equations

Fluids obey the general laws of continuum mechanics: conservation of mass, energy, and linear momentum. They can be written as mathematical equations once a representation for the state of a fluid is chosen. In the context of mathematics, there are two classical representations. One is the so-called Lagrangian representation, where the state of a fluid "particle" at a given time is described with reference to its initial position. The other representation (adopted throughout this book) is the so-called Eulerian representation, where at each time t and position $\mathbf{x}$ in space the state – in particular, the velocity $\mathbf{u}(\mathbf{x}, t)$ – of the fluid "particle" at that position and time is given.

In the Eulerian representation of the flow, we also represent the density $\rho(\mathbf{x}, t)$ as a function of the position $\mathbf{x}$ and time t. The conservation of mass is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1.1}$$

The conservation of momentum is expressed in terms of the acceleration $\boldsymbol{\gamma}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$:

$$\rho \gamma_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \quad i = 1, 2, 3. \tag{1.2}$$

Here $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2,3}$, componentwise in the 3-dimensional case. Moreover, $\mathbf{f} = (f_i, f_2, f_3)$ represents volume forces applied to the fluid.

The acceleration vector $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\mathbf{x}, t)$ of the fluid at position $\mathbf{x}$ and time t can be expressed, using purely kinematic arguments, by the so-called material derivative

$$\boldsymbol{\gamma} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}, \tag{1.3}$$

or, componentwise,

$$\gamma_i = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, \quad i = 1, 2, 3.$$

Inserting this expression into the left-hand side (LHS) of equation (1.2) yields the term $\rho(\mathbf{u} \cdot \nabla)\mathbf{u}$, which is the only nonlinear term in the Navier–Stokes equations; this term is also called the *inertial term*. The Navier–Stokes equations are among the very few equations of mathematical physics for which the nonlinearity arises not from the physical attributes of the system but rather from the mathematical (kinematical) aspects of the problem.

Further transformations of the conservation of momentum equation necessitate additional physical arguments and assumptions. Rheology theory relates the stress tensor to the velocity field for different materials through the so-called stress–strain law and other constitutive equations. Assuming the fluid is Newtonian, which is the case of interest to us, amounts to assuming that the stress–strain law is linear. More precisely, for Newtonian fluids the stress tensor is expressed in terms of the velocity field by the formula

$$\sigma_{ij} = \mu \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} + (\lambda \operatorname{div} \mathbf{u} - p)\delta_{ij}, \tag{1.4}$$

where $p = p(\mathbf{x}, t)$ is the pressure. Here, δ_{ij} is the Kronecker symbol and μ, λ are constants. The constant μ is called the shear viscosity coefficient, and $3\lambda + 2\mu$ is the dilation viscosity coefficient. For thermodynamical reasons, $\mu > 0$ and $3\lambda + 2\mu \geq 0$. Inserting the stress–strain law (1.4) into the momentum equation (1.2), we obtain

$$\rho \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right\} = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} - \nabla p + \mathbf{f}. \tag{1.5}$$

Equations (1.1) and (1.5) govern the motion of compressible Newtonian fluids such as the air at high speeds (Mach number larger than 0.5). If we also assume that the fluid is incompressible and homogeneous, then the density is constant in space and time: $\rho(\mathbf{x}, t) \equiv \rho_0$. In this case, the continuity equation is reduced to the *divergence-free* condition:

$$\operatorname{div} \mathbf{u} = 0. \tag{1.6}$$

Because the density is constant, we may divide the momentum equation (1.5) by ρ and consider the so-called kinematic viscosity $\nu = \mu/\rho_0$; we may then replace the pressure p and the volume force $\mathbf{f}$ by the kinetic pressure p/ρ_0 and the mass density of body forces $\mathbf{f}/\rho_0$, respectively. In doing so, and taking into consideration the

divergence-free condition (1.6), we obtain the *Navier–Stokes equations for a viscous, incompressible, homogeneous flow*:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.7a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.7b}$$

where, for notational simplicity, we represent the divergence of $\mathbf{u}$ by $\nabla \cdot \mathbf{u}$. For all pratical purposes, the density has actually been normalized to unity; even so, we may sometimes replace (1.7a) by (1.5), remembering then that $\nabla \cdot \mathbf{u} = 0$ and ρ is constant.

For more details on the physical aspects of fluid mechanics, we refer the reader to the classical books of Batchelor [1988] and Landau and Lifshitz [1971].

It is readily accepted that the Navier–Stokes equations govern the motion of common fluids such as air or water, so we are faced with the persistent challenging question of recovering from (1.7) such complex motions as that of smoke dispersion in the air and the turbulent flow of a river around a bridge pillar.

The flow of fluids at the microscopic level is governed by phenomena in the realm of statistical mechanics of fluids. The appropriate statistics is given by the solution of the Boltzmann equation. That equation represents the evolution of the governing distribution function, which is dependent on the position and velocity of the particles colliding with one another as a result of thermal excitation at any finite temperature. The collisions are described by an integral collision operator. In general, the collision operator represents simultaneous collisions among many particles, necessitating the use of a many-particle distribution. As such, it is very complicated and essentially impossible to evaluate precisely. Only in the case of dilute gases can one limit oneself to considering the evolution of a single-particle distribution and to binary collisions, since many body collisions are highly unlikely. In this idealized situation, the collision operator can be approximated by first-order and second-order spatial derivatives. The former is the familiar pressure gradient and the latter is the Laplacian operating on the velocity, multiplied by a constant known as the viscosity. With that approximation in hand, we can take the appropriate moments of the one-particle Boltzmann equation and so derive first the conservation of mass equation and second the conservation of momentum equation that we recognize as the NSE (when the incompressibility condition is a valid assumption).

Although such a derivation has been carried out for dilute gases, a corresponding exercise for liquids remains an open problem. This is because binary collisions play a relatively minor role in liquids, which are much denser than gases and hence feature collisions between clusters of particles. However, for practical reasons and lacking a better option, we use the Navier–Stokes equations with a simple constant viscosity as a reasonable model for liquid flows.

The origin of viscosity imposes a limit on the domain of validity of the Navier–Stokes equations. Thus phenomena on a length scale comparable to or smaller than the collision mean free path in air at atmospheric pressure (say, 10^{-3} cm) cannot be described by a continuum model such as the NSE. Subsequently we will learn about

some natural lengths that characterize the length scale region in which flow energy dissipation is dominated by viscous phenomena. It will be important then to be sure that we are still in the regime characterized by a continuum model of the flow. A similar cautionary remark applies to the amplitude of fluctuations in turbulent flows: once we are in a regime in which those fluctuations are comparable with thermally (finite temperature) induced fluctuations, the model based on Navier–Stokes equations ceases to be relevant.

Nondimensional Form of the Navier–Stokes Equations

It is sometimes convenient, both for physical discussions and mathematical transparency, to consider a nondimensional form of the conservation of momentum equation. For that purpose we introduce a reference length L_* and a reference time T_* for the flow, and we set

$$\mathbf{x} = L_*\mathbf{x}', \quad t = T_*t', \quad p = P_*p', \quad \mathbf{u} = U_*\mathbf{u}', \quad \mathbf{f} = \frac{L_*}{T_*^2}\mathbf{f}',$$

where $P_* = U_*^2$ and $U_* = L_*/T_*$ are a reference pressure and a reference velocity, respectively. By substitution into (1.7) we obtain for $\mathbf{u}'$, p', $\mathbf{f}'$ the same equation but with ν replaced by Re^{-1}, where Re is a nondimensional number called the *Reynolds number*:

$$\mathrm{Re} = \frac{L_*U_*}{\nu}. \tag{1.8}$$

The value of the Reynolds number depends on the choice of the reference length and velocity. Usually, if Ω (the domain occupied by the fluid) is bounded then L_* can be taken as the diameter of Ω or as some other large-scale length related to Ω, such as the width of a channel. The choice of U_* (and hence of T_*) depends on the type of forcing of the flow; it can be related to the forces applied at the boundary of Ω or to a pressure gradient, for example. Various choices of L_* and U_* can be appropriate for a given flow, leading to various definitions of the Reynolds number, but turbulent flows result for all appropriate choices when Re is large. How large depends to some extent on the shape of the domain occupied by the fluid. Once the shape of the domain Ω is fixed, however, rescalings in length (L_*) and velocity (U_*) and changes in viscosity (ν) affect the equations only through the single parameter Re.

Hence, different experiments may lead to the same nondimensional equations. For example, multiplying the velocity by 2 and dividing the diameter of the domain by 2 leaves the Reynolds number unchanged, so we can pass from one experiment to another; this is the Reynolds similarity hypothesis constantly used in mechanical engineering. At a given Reynolds number, flows remote from the boundaries of the domain Ω, irrespective of the latter's shape, are similar owing to some universality properties of turbulent flows. Moreover, with flows around blunt bodies (say, a sphere), as the body's radius increases and the flow velocity and/or viscosity is adjusted so as to maintain the Reynolds number constant, the flow throughout the

modified flow domain remains similar. That is what has made possible the design of aircraft by means of relatively small models tested in moderately sized wind tunnels.

In Chapter III, instead of the Reynolds number we will use another nondimensional number: the Grashof number (see Section 13 in Chapter II).

A heuristic argument illustrating the significance of the Reynolds number emerges by comparing the inertial and dissipation terms of the Navier–Stokes equations. The inertial term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ has dimension

$$\frac{U_*^2}{L_*},$$

while the dissipation term has dimension

$$\nu \frac{U_*}{L_*^2}.$$

The inertial term dominates when

$$\mathrm{Re} = \frac{L_* U_*}{\nu} \gg 1.$$

However, a much more subtle analysis that is valid at each length scale is made for the Kolmogorov theory of turbulence.

By setting $\mathrm{Re} = +\infty$ (i.e., $\nu = 0$), we obtain the case of inviscid flows. In this case, the divergence-free condition is retained but the momentum equation changes, resulting in the Euler equations for inviscid perfect fluids:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.9a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.9b}$$

Note that some of the difficulties encountered in studying turbulent behavior, a largely inviscid regime, arise because the transition from Euler's equations to the Navier–Stokes equations necessitates a change from a first-order system to a second-order one in space (∇ to Δ), which involves a singular perturbation.

2 Turbulence: Where the Interests of Engineers and Mathematicians Overlap

Principal substantive questions related to turbulence have been raised since the beginning of the twentieth century, and a large number of empirical and heuristical results were derived – motivated principally by engineering applications. This includes the work of Lamb [1957], mostly on addressing idealized inviscid flows; Prandtl [1904], on eddy viscosity and boundary layers; Taylor [1935, 1937], on viscous flows; and von Karman [1911, 1912], on the nature of the boundary layer.

At the same time, in mathematics there appears the pioneering work of Jean Leray [1933, 1934a,b] on the Navier–Stokes equations. Leray speculated that turbulence is

due to the formation of point or "line vortices" on which some component of the velocity becomes infinite.[1] To enable dealing with such a situation, he suggested the concept of weak, nonclassical solutions to the Navier–Stokes equations (1.7), and this has become the starting point of the mathematical theory of the Navier–Stokes equations to this day. We will consider this approach in Chapter II and beyond. It is noteworthy that, more generally, Leray's ideas serve also as the starting point for several important elements of the modern theory of partial differential equations. Even today, despite much effort, Jean Leray's conjecture concerning the appearance of singularities in 3-dimensional turbulent flows has been neither proved nor disproved. Let us mention, however, the result of Caffarelli, Kohn, and Nirenberg [1982] (see also Scheffer [1977]), which considerably extends an earlier result of Leray: Given the possibility that the singular points are a fractal set (assuming that such a set exists), the 1-dimensional Hausdorff measure of that set in space and time is 0. Hence the occurence of smooth line vortices is not possible, explaining our quotation marks around "line vortices." Nevertheless, for all physical purposes this powerful mathematical result leaves room for a tremendously complex set of singularities, and so we remain far from closing the issues raised by Leray's conjecture.

Before continuing with these historical notes, we remark in passing that engineers are not directly affected by such purely mathematical issues; rather, they want to calculate or measure certain physical quantities (forces, velocities, pressures, etc.). Here, however, beside the possible occurrence of singularities, another critical aspect of turbulence comes to mind: in a turbulent flow, many interesting quantities vary rapidly in time and cannot be readily measured. In practice, all that can be measured in laboratory experiments are averages (usually time averages). These averages are well-defined, reproducible quantities. This leads to the concept of ensemble averages underlying the conventional theory of turbulence, and to the concept of statistical solutions of the Navier–Stokes equations (1.7). It leads also to the idea of *ergodicity*, which is taken for granted by engineers. Loosely speaking, for all initial experimental conditions and for all sorts of reasonable ensemble averages, the experiments always yield the same measured results to within the accuracy of the measurements. We address here those questions of direct interest to engineers: the need for statistical solutions, the equivalence between ensemble averages and time averages (a question addressed in Chapter IV), and the so far unchallenged issue of the axiomatic nature of ergodicity.

We return to our brief overview of some highlights in the history of the studies of turbulent flows. It is impossible to explore here all the aspects of that history. Hence, with apologies to all whose important contributions are not mentioned here, we limit ourselves to those aspects of the history most relevant to the subject of this monograph.

[1] In fact, if such discontinuities occur then another question of physical nature needs to be raised concerning the validity of the Navier–Stokes equations themselves; indeed, at very short distances of order 10^{-3} cm (the collision mean free path of the particles), the fluid equations are no longer pertinent.

Turbulent flows have mystified people for ages, as evidenced for example by Leonardo da Vinci's sketches of the turbulent wakes downstream of some bridge columns. Beginning with careful experimental studies of flows under various experimental conditions (Reynolds [1883, 1895]) and with the subsequent formulation of the Navier–Stokes equations, turbulence became a subject of thorough scientific inquiry. For many years, two difficulties held the attention of various investigators. The first was a technical mathematical obstacle: the presence of the inertial term (a quadratic nonlinearity) precludes a straightforward use of the many available tools of perturbation methods. The structure of the equations demands that, at any given step in an approximation scheme, information from the next step is necessary. This had led to many attempts at formulating the so-called closure schemes, where at some step in the approximation sequence an assumption about the nature of the subsequent term is made, thereby terminating that sequence. Such an assumption, usually justified in terms of intuitive physical arguments, was then used to break the impasse in the approximation sequence. In principle, closure schemes by and large call for unprovable assumptions beyond those composing the basis for the Navier–Stokes equations. Some of the better-known closure schemes may be found in such texts as Tennekes and Lumley [1972], Leslie [1973], and Lesieur [1997], although further attempts (and controversies) in this area continue. As we shall find in the present work, the invention of the so-called inertial manifolds in the context of the rigorous theory of NSE (as well as of other nonlinear partial differential equations) opens the door to mathematically more soundly based schemes for computational approaches, offering an alternative to the conventional closure schemes.

The second obstacle to progress in the theory of turbulence was largely conceptual. Namely, how was it possible for a system described by perfectly deterministic equations to exhibit behavior that was undeniably statistical in nature? This aspect of turbulent flows, both from the experimental side and from the nascent theoretical side, is dealt with at length in the monumental work of Monin and Yaglom [1975]. Hopf [1952], followed by Foias and Prodi [1976] (see also Foias [1972, 1973, 1974]), studied an extension of Liouville's theorem that in principle yields the probability distribution function underlying the Navier–Stokes equation. Many of these efforts rested on the experimental and theoretical work of Taylor [1935, 1937] and von Karman and Howarth [1938], who clarified, on intuitive grounds, the nature of homogeneous isotropic turbulence. The simplifications resulting from the symmetries inherent in this idealized form of turbulence yielded the well-known von Karman–Howarth ordinary differential equation for the self-similar evolution of the two-point velocity correlation tensor. This idealization has also yielded Kolmogorov's theory for the spectrum of homogeneous isotropic turbulence in three dimensions (Kolmogorov [1941a,b]) (and later Batchelor's [1959] and Kraichnan's [1967] corresponding results for turbulence in two dimensions), a subject of the next section. All of these results were obtained without full understanding of the origin of the statistical nature of turbulence. A significant breakthrough occurred in the 1960s and 1970s with the discovery of stochastic instabilities in seemingly innocuous low-order ordinary

differential equations (Lorenz [1963]) and in some nonlinear difference equations (Feigenbaum [1980]). Subsequent research (Foias and Prodi [1976], Vishik and Fursikov [1977a,b, 1978], Foias and Temam [1979]) on dynamical systems governed by nonlinear partial differential equations revealed that such dynamical systems may reside, in finite-dimensional function spaces, on compact attractors that may be characterized by chaotic behavior.

It is now appropriate to reiterate a point hinted at earlier, namely, the essential need for careful mathematical analysis when dealing with nonlinear entities such as the Navier–Stokes equations. While much of our physical intuition serves us well in the domain of linear phenomena modeled adequately by linear differential and partial differential equations, it can fail us – with potentially disastrous consequences – in nonlinear domains. A fairly instructive example, outside the realm of this book but worth mentioning here, concerns modeling sonic flow transition as a boundary value problem rather than (and more correctly) as an initial value problem (Greenberg and Trève [1960]). Although this may appear to be unnecessary pedantry, it clearly makes a lot of difference in the context of, say, nuclear reactor safety (Bilicki et al. [1987]). Unlike the case in linear systems, in nonlinear systems small causes can lead to very large effects indeed, as well as to qualitative differences. Because nonlinear equations can have multiple, qualitatively different solutions (different basins of attraction), a small change in initial conditions can sometimes lead to radically different time-asymptotic behavior. An even more dramatic, counterintuitive example is the previously mentioned possibility of chaotic behavior in what at first sight seem to be innocent deterministic systems (Lorenz [1963], Feigenbaum [1980], Smale [1967]). Here is a class of problems in which necessarily limited computer "experiments" can lead to misleading conclusions about the behavior of a system as a function of the governing parameters. Only a thorough analysis of the system can reveal its true nature. Occasionally, such an analysis will reveal, even without detailed numerical computations, an unphysical aspect of the system (e.g., infinite energy density, decreasing entropy, or other pathologies), which is a clear alert to the flawed nature of the system model.

In this work we concentrate on those aspects of turbulent fluid flows that can be represented in terms of so-called Sobolev spaces – that is, a class of functions satisfying the given boundary conditions – and the given physical constraints, such as divergence-free (incompressible) flow. The various norms (i.e., various integrals of some seemingly abstract quantities) in these function spaces are in fact readily recognized as tangible physical quantities that are more or less readily accessible to direct experimental observation. The relationships among these norms, and the rules for their manipulations, reveal some aspects of the turbulent flows that justify many ad hoc interpretations and inspire insights derived from direct observations of turbulence while also revealing some hitherto unrealized ones. As such, these mathematical endeavors can serve to enlarge our intuitive horizons beyond the limits of linear theories and models.

3 Elements of the Theories of Turbulence of Kolmogorov and Kraichnan

Turbulent flows seem to display self-similar statistical properties at length scales smaller than the scales at which energy is delivered to the flow. Kolmogorov [1941a,b] argued that, at these scales, in three dimensions, the fluids display universal statistical features. Turbulent flow is conventionally visualized as a cascade of large eddies (large-scale components of the flow) breaking up successively into ever smaller sized eddies (fine-scale components of the flow; Onsager [1945]). Such a cascade, or flow of kinetic energy from large to small scales, is taken to occur in a regime at lengths sufficiently large for the effects of viscosity to be inconsequential. The apparent energy dissipation – that is, the removal of energy from one length scale to a smaller one – is solely due to the presence of the nonlinear (inertial) term in the Navier–Stokes equations. The energy dissipation rate $\epsilon = \nu\kappa_0^3|\nabla\mathbf{u}(\mathbf{x},t)|^2$ is assumed to be constant in space and time. A further essential assumption is that the cascade proceeds so that, at every length scale (or at every corresponding wavenumber), there is an equilibrium between energy flowing in from above to a given scale and that flowing out to a lower scale. Such a picture and the associated assumptions imply that, in this range of length scales (or this range of wavenumbers), the energy density at a given wavenumber can depend only on the energy dissipation rate ϵ and the wavenumber k itself. Then dimensional analysis alone yields $\mathcal{S}(\kappa) = \text{const.} \times \epsilon^{2/3}/\kappa^{5/3}$ for the energy density. Such a cascade process cannot continue to arbitrarily small length scales because, as the norm of the Laplacian operator increases with the decreasing length scale, eventually the effects of molecular dissipation begin to dominate the nonlinear inertial term. That length, denoted by ℓ_d, is the endpoint of the inertial range and the beginning of the dissipation range.

Let us determine ℓ_d. At each scale ℓ (or wavenumber $\kappa = \ell^{-1}$), we can define by dimensional analysis, through ϵ and ℓ, a natural time scale τ and speed u. Indeed, $\epsilon = \ell^2/\tau^3$ gives $\tau = (\ell^2/\epsilon)^{1/3}$ and $u = \ell/\tau = (\ell\epsilon)^{1/3}$. Now, the dissipation length ℓ_d is where the viscous term $\nu\Delta\mathbf{u}$ starts to dominate, on average, the inertial term. Hence,

$$\nu\Delta\mathbf{u} \sim \frac{\nu u}{\ell^2} \sim \frac{\nu}{\ell\tau} > (\mathbf{u}\cdot\nabla)\mathbf{u} \sim \frac{u^2}{\ell} \sim \frac{\ell}{\tau^2}.$$

Therefore,

$$\ell^2 < \nu\tau = \nu\left(\frac{\ell^2}{\epsilon}\right)^{1/3} \iff \ell^{4/3} < \left(\frac{\nu^3}{\epsilon}\right)^{1/3}$$

and

$$\ell_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \tag{3.1}$$

Kolmogorov thus inferred that, in 3-dimensional turbulent flows, the eddies of length size sensibly smaller than ℓ_d are of no dynamical consequence. As we said,

the length ℓ_d as defined by (3.1) is known as the Kolmogorov *dissipation length*. The corresponding wavenumber,

$$\kappa_d = \frac{1}{\ell_d} = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}, \tag{3.2}$$

is the Kolmogorov *dissipation wavenumber*.

The inertial range, within which inertial effects dominate, is the range $\ell_1 < \ell < \ell_d$, where $\ell_1 = L_1$ is the wavelength at which energy is injected in the flow. To each length ℓ in this range we can associate a Reynolds number $\mathrm{Re}_\ell = u\ell/\nu$; hence,

$$\mathrm{Re}_\ell^{3/4} = \ell\left(\frac{\epsilon}{\nu^3}\right)^{1/4}.$$

The largest of these Reynolds numbers obtained for $\ell =$ the Kolmogorov macro-scale length $L_* \simeq L_1$ is the Reynolds number Re of the flow. Hence, with (3.1),

$$\mathrm{Re} = \left(\frac{L_*}{\ell_d}\right)^{4/3}, \quad \text{or} \quad L_* = \mathrm{Re}^{3/4}\,\ell_d. \tag{3.3}$$

This relationship leads naturally to the heuristic estimate of the number of degrees of freedom in 3-dimensional flows, which is $\mathrm{Re}^{9/4}$. As we shall see, this heuristic estimate is actually an upper bound on the sufficient (but not necessary) number of degrees of freedom in 3-dimensional turbulent flows.

We now present a somewhat more elaborate derivation (but one that is still divorced from the Navier–Stokes equations) of the so-called Kolmogorov spectrum.

Let ϵ denote the average of the energy per unit mass. Then, according to the Kolmogorov theory, the length ℓ_d at which the turbulent eddies are rapidly annihilated by the viscosity should be a universal function of ϵ and the kinematic viscosity ν, namely:

$$\ell_d = f(\nu, \epsilon). \tag{3.4}$$

In particular, f should be independent of the choice of units for space and time. Thus, if we pass from $\mathbf{x}, t$ to $\mathbf{x}' = \xi\mathbf{x}$ and $t' = \tau t$ then we should still have

$$\ell_d' = f(\nu', \epsilon'). \tag{3.5}$$

Here ν' and ϵ' are not independent of ν and ϵ, and dimensional analysis yields

$$\ell_d' = \xi\ell_d, \quad \nu' = \frac{\xi^2}{\tau}\nu, \quad \epsilon' = \frac{\xi^2}{\tau^3}\epsilon; \tag{3.6}$$

that is,

$$\xi f(\nu, \epsilon) = f(\xi^2\tau^{-1}\nu, \xi^2\tau^{-3}\epsilon). \tag{3.7}$$

With the choices

$$\frac{\xi^2}{\tau} = \frac{1}{\nu} \text{ and } \frac{\xi^2}{\tau^3} = \frac{1}{\epsilon}, \quad (\text{i.e., } \tau = (\epsilon\nu)^{1/2} \text{ and } \xi = \epsilon^{1/4}/\nu^{3/4}),$$

the relation (3.7) becomes

$$f(\nu, \epsilon) = \frac{1}{\xi} f(1, 1) \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}.$$

Following Kolmogorov, one can also argue that the average energy per unit mass, $e_{\kappa,2\kappa}$, of the eddies of lengths between $\ell/2$ and ℓ (i.e., between the wavenumbers κ and 2κ, where $\kappa = 1/\ell$) should enjoy a similar universal property – namely,

$$e_{\kappa,2\kappa} = g(\epsilon, \kappa), \tag{3.8}$$

provided that $\kappa \ll \kappa_d$ (so that the effect of the viscosity can be neglected) and that κ is much larger than the wavenumber at which energy is pumped into the flow. Again the universality of g implies that

$$\xi^2 \tau^{-2} g(\epsilon, \kappa) = g(\xi^2 \tau^{-3} \epsilon, \xi^{-1} \kappa);$$

whence, upon taking $\xi = \kappa$ and $\tau = \epsilon^{1/3} \kappa^{2/3}$, one obtains

$$e_{\kappa,2\kappa} = c \frac{\epsilon^{2/3}}{\kappa^{2/3}}, \tag{3.9}$$

where $c = g(1, 1)$, a universal constant.

Consider now the Navier–Stokes equations with periodic boundary conditions. That is, we consider the solutions of equations (1.7) that are periodic in space with period L in each direction. Using Fourier series expansions (see Sections 2 and 5 in Chapter II for details), we can write

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}} \hat{\mathbf{u}}_{\mathbf{k}} e^{\kappa_1 i \mathbf{k} \cdot \mathbf{x}}, \quad \hat{\mathbf{u}}_{\mathbf{k}} \cdot \mathbf{k} = 0, \ \hat{\mathbf{u}}_{-\mathbf{k}} = \overline{\hat{\mathbf{u}}}_{\mathbf{k}}. \tag{3.10}$$

For $\mathbf{k}$ in (3.10), $\kappa = \kappa_1 |\mathbf{k}|$ is the corresponding wavenumber, where $\kappa_1 = 2\pi/L$. The lowest wavenumber is κ_1. The component of $\mathbf{u}$ with wavenumbers between κ' and κ'' is

$$\mathbf{u}_{\kappa',\kappa''} = \sum_{\kappa' \le \kappa < \kappa''} \hat{\mathbf{u}}_{\mathbf{k}} e^{\kappa_1 i \mathbf{k} \cdot \mathbf{x}}. \tag{3.11}$$

The energy per unit mass and the enstrophy (square of vorticity) per unit mass are, respectively,

$$|\mathbf{u}_{\kappa',\kappa''}|^2 = \sum_{\kappa' \le \kappa < \kappa''} |\hat{\mathbf{u}}_{\mathbf{k}}|^2 \tag{3.12}$$

and

$$\|\mathbf{u}_{\kappa',\kappa''}\|^2 = \kappa_1^2 \sum_{\kappa' \le \kappa < \kappa''} |\mathbf{k}|^2 |\hat{\mathbf{u}}_{\kappa}|^2. \tag{3.13}$$

Note that $\mathbf{u}_{\kappa_1,\infty} = \mathbf{u}$.

Physicists and engineers assume that, for $L \gg 1$, the time averages of

$$|\mathbf{u}_{\kappa',\kappa''}(t)|^2 \quad \text{and} \quad \kappa_1^2 \|\mathbf{u}_{\kappa',\kappa''}(t)\|^2 \tag{3.14}$$

exist (see Chapters IV and V). Note that the first average should be $e_{\kappa',\kappa''}$, the average of energy per unit mass of the eddies of linear size between $1/\kappa''$ and $1/\kappa'$. We denote these averages by

$$\langle |\mathbf{u}_{\kappa',\kappa''}(\cdot)|^2 \rangle \quad \text{and} \quad \kappa_1^2 \langle \|\mathbf{u}_{\kappa',\kappa''}(t)\|^2 \rangle. \tag{3.15}$$

Moreover, it is also assumed that these values can be viewed (at least when κ_1 is small) as integrals in the wavenumbers; that is,

$$\int_{\kappa'}^{\kappa''} \mathcal{S}(\kappa)\, d\kappa \quad \text{and} \quad \int_{\kappa'}^{\kappa''} \mathcal{S}_1(\kappa)\, d\kappa. \tag{3.16}$$

Comparing (3.12) and (3.13) with (3.15), we see that if (3.16) makes sense then one is led to the relation

$$\mathcal{S}_1(\kappa) \equiv \kappa^2 \mathcal{S}(\kappa). \tag{3.17}$$

The function $\mathcal{S}(\kappa)$ (≥ 0) is called the *energy spectrum* of the turbulent flow produced by the driving force $\mathbf{f}$ in (1.7). Also, the driving force is assumed to have no high-wavenumber components: $\mathbf{f} = \mathbf{f}_{\kappa_1,\bar{\kappa}}$, where $\bar{\kappa}$ is comparable in size with κ_1 (the lowest wavenumber).

So, according to Kolmogorov's theory, we have (see (3.9))

$$\int_{\kappa}^{2\kappa} \mathcal{S}(\chi)\, d\chi \sim c \frac{\epsilon^{2/3}}{\kappa^{2/3}}, \tag{3.18}$$

at least as long as

$$\bar{\kappa} \ll \kappa \ll \kappa_d. \tag{3.19}$$

Taking the derivative in (3.18) yields

$$\mathcal{S}(\kappa) - \mathcal{S}(2\kappa) \sim \frac{2c}{3} \frac{\epsilon^{2/3}}{\kappa^{5/3}},$$

whence

$$\mathcal{S}(\kappa) \sim \mathcal{S}(2^{m+1}\kappa) + \frac{2c}{3}\left(1 + \frac{1}{2^{5/3}} + \frac{1}{2^{10/3}} + \cdots + \frac{1}{2^{5m/3}}\right)\frac{\epsilon^{2/3}}{\kappa^{5/3}} \tag{3.20}$$

as long as $2^m\kappa \ll \kappa_d$. For turbulent flows, $\kappa_d \gg \kappa_1 \approx \bar{\kappa}$ and so we may take $m \gg 1$ in (3.20). Then

$$\mathcal{S}(\kappa) \sim C_K' \frac{\epsilon^{2/3}}{\kappa^{5/3}}, \tag{3.21}$$

where $C_K' = (2/3)c(1 - 2^{-5/3})^{-1}$. The form (3.21) for the energy spectrum is called the *Kolmogorov energy spectrum* of the turbulent flow. The constant C_K' is known as the *Kolmogorov constant* in energy space (there is a similar relation in which a constant C_K appears and takes the name of Kolmogorov constant in physical space; see (5.26) in Chapter V). The empirical value of C_K' is of the order of unity. The range of κ in (3.19) for which (3.21) holds is the Kolmogorov *inertial range.*

It must be noted that the estimate (3.21) is really a time average, as the amplitude of $\mathcal{S}(\kappa)$ fluctuates wildly in time. Furthermore, it is only an approximation. In reality, for a turbulent flow in a bounded domain, intermittency effects in the energy

dissipation rate ϵ result in small but measurable corrections to the simple expression (3.21) (see e.g. Kolmogorov [1962] and Novikov and Stewart [1964]). These corrections depend on the size (say, L_*) of Ω, thus destroying – to some extent – the universality of turbulence.

As seen in the preceding, the arguments leading up to (3.21) are clearly divorced from the NSE itself and are applicable solely to turbulent flows in three dimensions. For flows in two dimensions (which, as stated earlier, are amenable to deep analysis), we must turn to a phenomenological theory proposed by Batchelor [1959] and Kraichnan [1967]. That theory is in the spirit of Kolmogorov's approach but does not parallel it because the physical situation is quite different; hence we offer a separate exposition.

Two-dimensional flows are not commonly encountered in nature. Examples that do come to mind are thin liquid films and (to within some approximation) the atmospheric layer on the surface of the Earth – although clearly the significant phenomena (e.g., weather and climate changes) occur on scales within which the finite thickness of that layer must be taken explicitly into account. However, some firm mathematical results derived in the study of 2-dimensional flows appear to carry over to some 3-dimensional flows, so it is instructive to follow what can be learned about 2-dimensional flows. Moreover, further advances in functional analysis of the Navier–Stokes equations in three dimensions may yield the necessary tools for solving some critical open problems. For now we turn to a summary of Kraichnan's work on the phenomenological theory of 2-dimensional turbulence.

We limit ourselves here to fluid flows in the plane, although much of the theory could carry over to more general 2-dimensional manifolds. The principal physical difference between 2-dimensional and 3-dimensional flows is that, in the 2-dimensional case, the vorticity (i.e., the curl of the velocity) has only one component – in the direction normal to the plane of the flow. This imposes a severe constraint on the kinematics and the dynamics of the turbulence. For instance, in addition to the conservation of energy, the flow must conserve enstrophy, that is, the integral of the square of the vorticity over the flow domain must be constant. The nonlinear interactions may be viewed in wavenumber (Fourier) space as three-wave interactions. They cannot simultaneously satisfy the two conservation principles. Hence, the energy cascade cannot coexist with the enstrophy cascade; they must occur in distinct portions of the wavenumber domain. As a consequence, in the turbulent regime (large Reynolds number) for the 2-dimensional case, there are two contiguous ranges: for wavenumbers lower than that at which the forcing of the flow is introduced there is an inverse energy cascade, with small eddies coalescing into larger ones and with the cascade terminating at a wavenumber determined by the size of the flow domain. The spectrum of the energy, $\mathcal{S}(\kappa)$, in that domain is the same as the Kolmogorov spectrum. Toward the higher wavenumbers, the principal cascading entity is the enstrophy – that is, the successive breakup of the vortices into ever smaller ones, with the attendant enstrophy dissipation rate η resulting from nonlinear interactions and not being affected by the molecular viscosity above the *Kraichnan cutoff length*. According to

Kraichnan, the two portions of the inertial range (i.e., the inverse cascade and the enstrophy cascade) cannot overlap.

Using arguments based on dimensional analysis along the lines followed by Kolmogorov in the 3-dimensional regime, we can now determine the energy spectrum in the enstrophy cascade region as well as the Kraichnan cutoff wavenumber. Note first that the enstrophy dissipation rate has the dimension of (time)$^{-3}$. Assuming that, in the enstrophy cascade range, the energy spectrum depends only on η and the wavenumber κ, we then find that $\mathcal{S}(\kappa) = \text{const.} \times \eta^{2/3}/\kappa^3$. Similarly, it follows from dimensional considerations that the Kraichnan cutoff wavenumber is given by

$$\kappa_\eta = \left(\frac{\eta}{\nu^3}\right)^{1/6}. \tag{3.22}$$

An extended treatment of turbulent flows in two dimensions is presented in Section 5 of Chapter IV.

4 Function Spaces, Functional Inequalities, and Dimensional Analysis

The mathematical theory of the Navier–Stokes equations is based on the use of function spaces, which are at the heart of the modern theory of partial differential equations. A formal presentation of the needed tools appears in Chapter II and thereafter as needed. However, in this section we give an informal introduction and emphasize that these spaces are not merely inventions of mathematicians; rather, they are strongly related to the physics of the problem.

The Fundamental Function Spaces

Consider the domain Ω occupied by the fluid; Ω is a domain of $\mathbb{R}^3$, which could be the whole space $\mathbb{R}^3$ in certain idealized cases. The first natural function space is $L^2(\Omega)$, the space of square integrable functions on Ω; we also have $L^2(\Omega)^3$, the space of square integrable vector fields on Ω. These spaces are endowed with a scalar product, which we denote by

$$(u, v) = \int_\Omega u(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x}$$

in the scalar case and, similarly,

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}$$

in the vector-valued case. To these scalar products correspond the following norms[2] (mean square norms):

$$|u| = \left(\int_\Omega |u(\mathbf{x})|^2\, d\mathbf{x}\right)^{1/2}, \qquad |\mathbf{u}| = \left(\int_\Omega |\mathbf{u}(\mathbf{x})|^2\, d\mathbf{x}\right)^{1/2}.$$

[2] We will use the same notation $|\cdot|$ for the Euclidean norm in $\mathbb{R}^2$ and $\mathbb{R}^3$ (and even $\mathbb{C}^2$ and $\mathbb{C}^3$), and also for various L^2 norms. This abuse of notation, for purposes of simplification, does not lead to any confusion once the context is taken into account.

The inner product and the corresponding norm are also related by the so-called Cauchy–Schwarz inequality:

$$|(u,v)| = \left| \int_\Omega u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} \right| \le \left(\int_\Omega |u(\mathbf{x})|^2\,dx \right)^{1/2} \left(\int_\Omega |v(\mathbf{x})|^2\,dx \right)^{1/2} = |u||v| \tag{4.1}$$

for all u, v in $L^2(\Omega)$. Similarly, for any $\mathbf{u}, \mathbf{v} \in L^2(\Omega)^3$, we have

$$|(\mathbf{u},\mathbf{v})| \le |\mathbf{u}||\mathbf{v}|. \tag{4.2}$$

Now, given the velocity vector field $\mathbf{u}$ of the fluid in Ω,

$$\mathbf{u}\colon \mathbf{x} \in \Omega \mapsto \mathbf{u}(\mathbf{x}) \in \mathbb{R}^3,$$

we see that the square of the L^2-norm, $|\mathbf{u}|^2$, is merely twice the kinetic energy of the flow (assuming that the density of the fluid has been normalized to unity):

$$e(\mathbf{u}) = \frac{1}{2}\int_\Omega |\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} = \frac{1}{2}|\mathbf{u}|^2.$$

Without entering into the details of measure theory, we recall that $L^2(\Omega)$ and $L^2(\Omega)^3$ are Hilbert spaces for these scalar products and norms. Also, for $L^2(\Omega)$ (and the same is true for $L^2(\Omega)^3$), the following characterizaton holds:[3]

> $u \in L^2(\Omega)$ *if and only if* there exists a sequence of smooth functions u_n, compactly supported in Ω, such that $|u_n|$ (or $e(u_n)$) remains bounded and u_n is converging to u (in the distribution sense) as $n \to \infty$. (4.3)

Most of the spaces that we will consider are derived from the space $L^2(\Omega)$ and from another space that we will now introduce, the so-called Sobolev space $H^1(\Omega)$.

In Chapter II we will address the concept of enstrophy of a fluid velocity $\mathbf{u} = (u_1, u_2, u_3)$, namely,

$$E(\mathbf{u}) = \sum_{i,j=1}^{3} \int_\Omega \left| \frac{\partial u_i}{\partial x_j} \right|^2 d\mathbf{x}.$$

By comparison with (4.3), a natural question is the following:

> What can we say of a sequence of smooth velocity vector fields $\mathbf{u}_n$ such that $E(\mathbf{u}_n)$ remains bounded? (4.4)

If Ω is bounded,[4] then one can prove that u_n contains one (or more) subsequence(s) that converge in the distribution sense to some limit $\mathbf{u}$. This vector function $\mathbf{u} = (u_1, u_2, u_3)$ is in $L^2(\Omega)^3$, as are its (distributional) first derivatives

$$\frac{\partial u_i}{\partial x_j}, \quad i, j = 1, 2, 3;$$

[3] The functions u_n in this characterization of $L^2(\Omega)$ are defined, say, by some kind of approximation procedure.

[4] If Ω is not bounded, then we should also require that $e(\mathbf{u}_n)$ remain bounded as well.

in fact,

$$E(\mathbf{u}) \leq \sup_{n\in\mathbb{N}} E(\mathbf{u}_n).$$

We say that such a function $\mathbf{u}$ belongs to the Sobolev space $H^1(\Omega)^3$ (with a similar definition for $H^1(\Omega)$ in the scalar case). In some sense:

> The space $L^2(\Omega)^3$ consists of all the vector fields $\mathbf{u}$ with finite kinetic energy, and the space $H^1(\Omega)^3$ consists of all the vector fields $\mathbf{u}$ with finite enstrophy. (4.5)

From the mathematical point of view, the Sobolev spaces $H^1(\Omega)$ and $H^1(\Omega)^3$ are Hilbert spaces for the following inner products and norms (see Chapter II):

$$((u,v))_1 = \frac{1}{L^2}\int_\Omega u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} + \sum_{j=1}^{3}\int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\,d\mathbf{x},$$

$$((\mathbf{u},\mathbf{v}))_1 = \frac{1}{L^2}\int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x} + \sum_{i,j=1}^{3}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x},$$

$$\|u\|_1 = [((u,u))_1]^{1/2}, \qquad \|\mathbf{u}\|_1 = [((\mathbf{u},\mathbf{u}))_1]^{1/2},$$

where L is a typical length (e.g., the diameter of Ω). In nondimensional variables, $L = 1$. As with the space $L^2(\Omega)$ and any other Hilbert space, we have the Cauchy–Schwarz inequality, which in this context reads

$$|((u,v))_1| \leq \|u\|_1\|v\|_1, \qquad |((\mathbf{u},\mathbf{v}))_1| \leq \|\mathbf{u}\|_1\|\mathbf{v}\|_1 \tag{4.6}$$

for all u, v in $H^1(\Omega)$ and all $\mathbf{u}, \mathbf{v}$ in $H^1(\Omega)^3$.

Most function spaces that we consider are derived from these two physically obvious spaces, the space $L^2(\Omega)^3$ of finite kinetic energy and the space $H^1(\Omega)^3$ of finite enstrophy. For instance, two central spaces V and H appear throughout the book; assuming for simplicity that Ω is bounded, a mathematically rigorous and physically intuitive definition of the spaces V and H is as follows:

> V is made up of all the limit points (in the distributional sense) of all the possible sequences of smooth vector fields $\mathbf{u}_n$ which are divergence-free, which satisfy the boundary conditions of the problem, and whose enstrophy remains bounded; that is, $E(\mathbf{u}_n) \leq$ const. $< \infty$. (4.7)

The space H is defined in a similar way, replacing the boundedness of the enstrophy by the boundedness of the kinetic energy: $e(\mathbf{u}_n) \leq$ const. $< \infty$. More details are given in Chapter II.

Functional Inequalities

The functions belonging to the space $H^1(\Omega)$ (and to other related spaces) satisfy certain inequalities, which are called Sobolev inequalities in the mathematical literature.

We make extensive use of these inequalities in the course of this book. Some of them are proven by interpolation, others by appropriate direct methods (e.g., Gagliardo–Nirenberg's, Agmon's, Ladyzhenskaya's, and Poincaré's inequalities). Our objective here is twofold:

(i) to prove the Ladyzhenskaya and the Poincaré inequalities; and
(ii) to emphasize the *physical invariance by dilatation* or *by change of scale* of all such inequalities.

The Ladyzhenskaya inequality may be described as follows. For a smooth, compactly supported scalar function u in $\mathbb{R}^2$, we have

$$\int_{\mathbb{R}^2} |u(\mathbf{x})|^4 \, d\mathbf{x} \leq \left(\int_{\mathbb{R}^2} |u(\mathbf{x})|^2 \, d\mathbf{x} \right) \left(\int_{\mathbb{R}^2} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \right). \tag{4.8}$$

In $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} |u(\mathbf{x})|^4 \, d\mathbf{x} \leq \left(\int_{\mathbb{R}^3} |u(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \right)^{3/2}. \tag{4.9}$$

In (4.8) and (4.9) – and in (4.10), (4.11), and (4.12) – the smoothness of the functions u and g is assumed for the sake of simplicity, since these inequalities are valid for more general (less smooth) functions.

Remark 4.1 Note the difference between space dimensions 2 and 3. It is this very discrepancy between the two cases that induces many of the difficulties in the mathematical theory of the Navier–Stokes equations in space dimension 3.

That no dimensional constant appears in the RHS of (4.8) or (4.9) is due to the fact that these inequalities are invariant by dilatation, or homothety ($\mathbf{x} \mapsto \lambda \mathbf{x}$), or (in physical terms) that both sides of these inequalities have the same dimension:

$$U^4 L^2 \sim U^2 L^2 \left(\frac{U^2 L^2}{L^2} \right) \quad \text{for (4.8)},$$

$$U^4 L^3 \sim (U^2 L^3)^{1/2} \left(\frac{U^2 L^3}{L^2} \right)^{3/2} \quad \text{for (4.9)}.$$

Remark 4.2 This invariance of the inequality by dilatation (or homogeneity) is common to many functional inequalities. However, the lack of a multiplicative constant in the RHS of (4.8) and (4.9) is not usual. In most cases, we know (we can prove) the existence of a multiplicative constant in the right-hand side of the inequalities, which in some cases may be obtained explicitly. In general, however, such constants can be taken as of order unity.

We present now the proof of (4.8). The proof of (4.9) follows a similar idea and uses (4.8). We start with the following inequality (of Agmon's type) in space dimension 1:

$$|g(x)|^2 \leq \left(\int_{\mathbb{R}} |g(\xi)|^2 \, d\xi \right)^{1/2} \left(\int_{\mathbb{R}} |g'(\xi)|^2 \, d\xi \right)^{1/2} \quad \text{for all } \ x \in \mathbb{R}, \tag{4.10}$$

for any smooth function g with compact support in $\mathbb{R}$. Since g has compact support, there exists $L > 0$ sufficiently large such that $g(x)$ vanishes for x outside $(-L, L)$. Then we can write

$$g(x)^2 = 2\int_{-L}^{x} g(\xi)g'(\xi)\,d\xi \le 2\int_{-L}^{x} |g(\xi)||g'(\xi)|\,d\xi$$

and

$$g(x)^2 = -2\int_{x}^{L} g(\xi)g'(\xi)\,d\xi \le 2\int_{x}^{L} |g(\xi)||g'(\xi)|\,d\xi.$$

Adding both identities, we obtain

$$g(x)^2 \le \int_{-L}^{L} |g(\xi)||g'(\xi)|\,dx = \int_{\mathbb{R}} |g(\xi)||g'(\xi)|\,dx.$$

Then, using the Cauchy–Schwarz inequality, we find (4.10).

We now prove the 2-dimensional Ladyzhenskaya inequality (4.8). We use the Agmon inequality (4.10) twice, once in each space direction. We have

$$\begin{aligned} u(x_1, x_2)^4 &= u(x_1, x_2)^2 u(x_1, x_2)^2 \\ &\le \left[\left(\int_{\mathbb{R}} |u(\xi_1, x_2)|^2\,d\xi_1\right)\left(\int_{\mathbb{R}} |u_{\xi_1}(\xi_1, x_2)|^2\,d\xi_1\right)\right. \\ &\qquad \left.\left(\int_{\mathbb{R}} |u(x_1, \xi_2)|^2\,d\xi_2\right)\left(\int_{\mathbb{R}} |u_{\xi_2}(x_1, \xi_2)|^2\,d\xi_2\right)\right]^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} u(x_1, x_2)^4\,dx_1\,dx_2 &\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |u(\xi_1, x_2)|^2\,d\xi_1 \int_{\mathbb{R}} |u_{\xi_1}(\xi_1, x_2)|^2\,d\xi_1\right)^{1/2} dx_2 \\ &\qquad \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |u(x_1, \xi_2)|^2\,d\xi_2 \int_{\mathbb{R}} |u_{\xi_2}(x_1, \xi_2)|^2\,d\xi_2\right)^{1/2} dx_1. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} u(x_1, x_2)^4\,dx_1\,dx_2 \\ &\quad \le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |u(\xi_1, x_2)|^2\,d\xi_1\,dx_2\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |u_{\xi_1}(\xi_1, x_2)|^2\,d\xi_1\,dx_2\right)^{1/2} \\ &\qquad \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |u(x_1, \xi_2)|^2\,d\xi_2\,dx_1\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |u_{\xi_2}(x_1, \xi_2)|^2\,d\xi_2\,dx_1\right) \\ &\quad \le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |u(x_1, x_2)|^2\,dx_1\,dx_2\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1, x_2)|^2\,dx_1\,dx_2\right), \end{aligned}$$

which proves (4.8). As mentioned earlier, the proof of (4.9) follows a similar idea: one first writes $u(x_1, x_2, x_3)^4$ as a product of two squares; then one applies the Agmon inequality (4.10) with respect to one variable and the 2-dimensional Ladyzhenskaya inequality (4.8) in the remaining two variables.

With little effort we can now prove another fundamental (often used) inequality: the Poincaré inequality.[5] We assume that u is smooth and that it vanishes outside a bounded set; in fact, it suffices for u to vanish outside a slab, say $-L/2 < x_1 < L/2$. In space dimension 2 (the proof is the same in any space dimension), we write (4.10) in direction x_1 with x_2 fixed:

$$u(x_1, x_2)^2 \le \left(\int_{\mathbb{R}} |u(\xi_1, x_2)|^2 \, d\xi_1 \right)^{1/2} \left(\int_{\mathbb{R}} \left| \frac{\partial u}{\partial \xi_1}(\xi_1, x_2) \right|^2 d\xi_1 \right)^{1/2}.$$

Observing that the RHS of this inequality does not depend on x_1, we integrate with respect to x_1, from $-L/2$ to $L/2$, and find

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x_1, x_2)|^2 \, dx_1 \, dx_2 \\
&\le L \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(\xi_1, x_2)|^2 \, d\xi_1 \right)^{1/2} \left(\int_{\mathbb{R}} \left| \frac{\partial u}{\partial \xi_1}(\xi_1, x_2) \right|^2 d\xi_1 \right)^{1/2} dx_2 \\
&\le \text{(after using the Cauchy–Schwarz inequality and renaming the dummy variable)} \\
&\le L \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x_1, x_2)|^2 \, dx_1 \, dx_2 \right)^{1/2} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x_1}(x_1, x_2) \right|^2 dx_1 \, dx_2 \right)^{1/2}.
\end{aligned}$$

This implies the Poincaré inequality

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x_1, x_2)|^2 \, dx_1 \, dx_2 \right)^{1/2} \le L \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x_1}(x_1, x_2) \right|^2 dx_1 \, dx_2 \right)^{1/2}, \tag{4.11}$$

as well as the following form, which is used more often:

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x_1, x_2)|^2 \, dx_1 \, dx_2 \right)^{1/2} \le L \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u(x_1, x_2)|^2 \, dx_1 \, dx_2 \right)^{1/2}. \tag{4.12}$$

The Poincaré inequality in higher dimensions can be proved similarly. In three dimensions, the form that we will often use reads

$$\left(\int_{\mathbb{R}^3} |u(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \le L \left(\int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}. \tag{4.13}$$

Remark 4.3 (cf. Remark 4.2) Note that a coefficient L (with dimension of length) appears in the RHS of (4.11) and (4.12). This length is given in the assumption on u (it vanishes outside $-L/2 < x_1 < L/2$), and both sides of (4.11) and of (4.12) have the same dimension, namely UL.

Another frequently used version of the Poincaré inequality (which we state without proof) relates to functions u defined on a bounded domain Ω whose average on Ω vanishes. We will use this inequality in space dimension $d = 2$ or 3, but for any space dimension d it reads as follows: There exists a constant $c = c(\Omega)$ such that

[5] More precisely, we prove one of the forms of this inequality; another form (valid for space-periodic functions) is given later in this section.

$$\int_\Omega |u(x_1,\dots,x_d)|^2\,dx_1\dots dx_d \le c(\Omega)\int_\Omega |\nabla u(x_1,\dots,x_d)|^2\,dx_1\dots dx_d \qquad (4.14)$$

for all functions u such that

$$\int_\Omega u(x_1,\dots,x_d)\,dx_1\dots dx_d = 0. \qquad (4.15)$$

This inequality will be frequently used for space-periodic functions with zero average on the period Ω. Note that the constant $c(\Omega)$, which is not easy to determine, has dimension L^2 (square of a length; see Remark 4.3). In the periodic case, we write this constant more explicitly as $c(\Omega) = \tilde{c}(\Omega)L^2$, where L is the smallest period. In this case, the constant $\tilde{c}(\Omega)$ depends only on the "shape" of Ω, in the sense that it is invariant under dilatation; see Remark 4.2.

More Inequalities

The methods of functional analysis employed throughout this volume rely heavily on the use of some relatively simple and well-known inequalities, as well as on more sophisticated ones. For the convenience of the reader, we list here (without proof) all inequalities in the first category. We shall then also list those in the second category – namely, the Sobolev inequalities and some of their variants, which extend (4.8) and (4.9) in various ways.

Schwarz's inequality:

$$ab \le \frac{1}{2}\left(\varepsilon a^2 + \frac{b^2}{\varepsilon}\right) \qquad (4.16)$$

for all real numbers a, b and all $\varepsilon > 0$.

Young's inequality:

$$ab \le \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \qquad (4.17)$$

for all $a, b > 0$ and all $1 < p < \infty$, with $p' = p/(p-1)$ (i.e., $1/p + 1/p' = 1$). Also,

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{\varepsilon^{1/(p-1)}p'}b^{p'} \qquad (4.18)$$

for all a, b, p, p' as before and all $\varepsilon > 0$.

Hölder's inequality:

$$\int_\Omega u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} \le \left(\int_\Omega |u(\mathbf{x})|^p\,d\mathbf{x}\right)^{1/p}\left(\int_\Omega |v(\mathbf{x})|^{p'}\,d\mathbf{x}\right)^{1/p'} \qquad (4.19)$$

for all measurable functions u and v for which the right-hand side is finite. Here, $1 < p < \infty$, $p' = p/(p-1)$, and Ω is an arbitrary open set in $\mathbb{R}^d$, $d \in \mathbb{N}$. Also,

$$\int_\Omega u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} \le \sup_{\mathbf{x}\in\Omega}|u(\mathbf{x})| \int_\Omega |v(\mathbf{x})|\,d\mathbf{x}. \qquad (4.20)$$

A weaker form of (4.20) involving the essential supremum of u appears in the sequel. In addition to the inequalities just listed, we use extensively the Poincaré inequalities