Navier–Stokes Equations and Turbulence

This book aims to bridge the gap between practicing mathematicians and the practitioners of turbulence theory. It presents the mathematical theory of turbulence to engineers and physicists as well as the physical theory of turbulence to mathematicians. The book is the result of many years of research by the authors, who analyze turbulence using Sobolev spaces and functional analysis. In this way the authors have recovered parts of the conventional theory of turbulence, deriving rigorously from the Navier–Stokes equations what had been arrived at earlier by phenomenological arguments.

The mathematical technicalities are kept to a minimum within the book, enabling the discussion to be understood by a broad audience. Each chapter is accompanied by appendices that give full details of the mathematical proofs and subtleties. This unique presentation should ensure a volume of interest to mathematicians, engineers, and physicists.

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Contents

Preface

This monograph is an attempt to address the theory of turbulence from the points of view of several disciplines. The authors are fully aware of the limited achievements here as compared with the task of understanding turbulence. Even though necessarily limited, the results in this book benefit from many years of work by the authors and from interdisciplinary exchanges among them and between them and others. We believe that it can be a useful guide on the long road toward understanding turbulence.

One of the objectives of this book is to let physicists and engineers know about the existing mathematical tools from which they might benefit. We would also like to help mathematicians learn what physical turbulence is about so that they can focus their research on problems of interest to physics and engineering as well as mathematics. We have tried to make the mathematical part accessible to the physicist and engineer, and the physical part accessible to the mathematician, without sacrificing rigor in either case. Although the rich intuition of physicists and engineers has served well to advance our still incomplete understanding of the mechanics of fluids, the rigorous mathematics introduced herein will serve to surmount the limitations of pure intuition. The work is predicated on the demonstrable fact that some of the abstract entities emerging from functional analysis of the Navier–Stokes equations represent real, physical observables: energy, enstrophy, and their decay with respect to time.

Beside this didactic objective, one of our scientific goals – in this book and in its underlying research – was to see what we can learn about the physical properties of turbulence using Sobolev spaces and the functional analysis methods that are based on them. As we subsequently show, these spaces – which seem to be abstract mathematical inventions – are in fact representations of observable physical quantities. In this way we have recovered several parts of the conventional theory of turbulence, deriving rigorously from the Navier–Stokes equations (NSE) what had been arrived at earlier by phenomenological arguments (Kolmogorov [1941a,b]), but in addition we derive new results. We have shown that the conventional estimate of the number of degrees of freedom in homogeneous, isotropic turbulence (viz., (Reynolds number)$^{9/4}$) is at best an upper bound on the number of degrees of freedom needed for numerical simulations of real flows. We have also provided a rigorous, mathematical way to avoid the common underlying assumption of the ergodicity of turbulent flows. In fact, we show that (in a suitable sense) time averages of various turbulent flow properties equal the related ensemble averages with respect to adequate statistical solutions; we have also found a means for removing the high-wavenumber components of the flow in such a way as to yield an effective viscosity, while providing a rough upper bound on the error committed relative to the true solution of the flow equation.

Another task, the second scientific objective of this book, was to make the connection between three of the classical approaches to turbulence: the Navier–Stokes equations; the dynamical systems approach (following the work and ideas of Lorenz [1963], Smale [1967], and Ruelle and Takens [1971]); and the conventional statistical theory of turbulence (following the works and ideas of Kolmogorov [1941a,b, 1962], Batchelor [1959], Kraichnan [1967], and others – e.g., Landau and Lifshitz [1971] and Monin and Yaglom [1975]). Before the research underlying the material presented here, these classical approaches evolved largely independently. In particular, the conventional theory of turbulence is based mostly on dimensional phenomenological arguments that traditionally make little reference to the NSE (see Tennekes and Lumley [1972]). However, we believe it is useful and instructive to show that many known results can be directly derived from the Navier–Stokes equations. We develop those connections to the widest possible extent.

The level of mathematical preparation necessary for understanding this material is an elementary knowledge of partial differential equations and their solutions in terms of eigenfunction expansions. Terms and concepts beyond that level are presented in detail as needed. Also included is a brief tutorial on Sobolev spaces and inequalities. To aid readers unfamiliar with some useful classical inequalities, they are presented (without proof) in Chapter I along with the tutorial.

Mathematically oriented readers are assumed to be familiar with elementary physics and continuum mechanics, including such principles as conservation of momentum and energy and the relationship between stress and strain. For their benefit, Chapter I contains also a short tutorial on the Kolmogorov (conventional) theory of turbulence.

One of the unresolved difficulties encountered in this monograph is due to limitations in the present stage of the mathematical theory of the NSE; the theory is fairly complete in the 2-dimensional case but still incomplete in dimension 3. Thus, while we realize that natural turbulence is usually 3-dimensional, here we sometimes emphasize 2-dimensional flows, which are fully within the grasp of modern methods of functional analysis.

The word *turbulence* has different meanings to different people, which indicates that turbulence is a complex and multifaceted phenomenon. For mathematicians, outstanding problems revolve around the Navier–Stokes equations (such as well-posedness and low-viscosity behavior, especially in the presence of walls or singular vortices). For physicists, major questions include ergodicity and statistical behavior as related to statistical mechanics of turbulence. Engineers would like responses to questions simple to articulate but amazingly difficult to answer: What are the heat transfer properties of a turbulent flow? What are the forces applied by a fluid to its boundary (be it a pipe or an airfoil)? To others pursuing the dynamical system approach, of interest is the large time behavior of the flow. Another ambitious question for engineers is the control of turbulence (to either reduce or enhance it), which is already within reach. Finally, a major goal in turbulence research – of interest to all and toward which progress is constantly made – is trustworthy and reliable computation of turbulent flows (see e.g. Orszag [1970] and Ferziger, Mehta, and Reynolds [1977]).

We do not address here any computational aspects, although this problem is very much present in our thoughts; neither do we address control problems, nor most of the practical engineering problems (see Schlichting [1960]). After the introductory and tutorial Chapter I, the core of the book consists of four chapters, Chapters II–V. Each of them, in addressing a particular topic, could actually be developed into a whole independent volume. Chapter II summarizes some classical and some more recent aspects of the mathematical theory of the Navier–Stokes equations – namely, their formulation and well-posedness. We start by presenting the physical background of the mathematical theory, introducing kinetic energy and enstrophy, conservation of kinetic energy, and the Helmholtz–Leray decomposition of vector fields. We present function spaces, the spaces of finite kinetic energy and finite enstrophy vector functions, as well as some additional related abstract spaces. After recalling the weak formulation of the NSE, a starting point of their mathematical theory going back to the work of Jean Leray in the early 1930s, we recall the main theorems of existence, uniqueness, and regularity of solutions. Then we describe analyticity properties of the solutions; first, analyticity in time, which is sometimes related to intermittency (a question briefly addressed in Sections 6.2 and 6.3 of Chapter V); and second, analyticity in space and time (Gevrey class regularity), which is related in the space-periodic case to the decline of Fourier coefficients of the solution. Finally, we briefly discuss the no-slip case with moving boundaries and establish properties of the rate of dissipation of flows.

Chapter III revolves around the idea (hinted at long ago by Landau and Lifshitz) that, in the permanent regime, turbulent flows as solutions of NSE are finite-dimensional. This concept, which in fact follows easily from the Kolmogorov approach to turbulence, was novel in its time; by now it has been substantiated in many different ways and extended as well to other equations modeling other physical phenomena. In Chapter III we discuss finite dimensionality of turbulent flows in the context of determining modes and nodes, showing that such flows are fully determined by either a finite (sufficiently large) number of modes or a finite number of observation points (nodes). We discuss also the large time behavior in the context of attractors and show finite dimensionality of attractors; all these dimensions are physically relevant and related to the Landau–Lifshitz estimates. We briefly discuss approximate inertial manifolds, the initial point for multilevel numerical algorithms under development; in some sense, these algorithms produce in time what multigrid or wavelet methods produce in space. Chapter IV comes closest to the issue of ergodicity. We introduce, in space dimensions 2 and 3, stationary statistical solutions and relate them to the limits of time averages. We consider also the corresponding invariant measure and relate it to the attractor that carries it. We then apply these tools to the study of the cascade processes in turbulent flows.

Finally, in Chapter V, we study the concept of statistical solutions, the evolution of the probability distribution of the flow, and homogeneous flows. We start by introducing the time-dependent statistical solutions on bounded domains. Then we introduce the (space-invariant) homogeneous statistical solutions for space-periodic flows and flows in the whole spaces. The Reynolds averaged equations are introduced, and we then discuss self-similar homogeneous statistical solutions (SSHSS); we introduce a 2-parameter family of such solutions from which, on the one hand, we resolve a paradox on SSHSS pointed out by Hopf [1952] and, on the other hand, we recover and complete some elements of the conventional theory of turbulence. For instance, we show how the Kolmogorov spectrum follows naturally from NSE and how the intermittency of turbulent flows is related to the fractal nature (see Novikov and Stewart [1964] and Mandelbrot [1982]) of energy dissipation in 3-dimensional flows.

As with all interdisciplinary work, it is not easy to write a book that is readable by (and of equal interest to) people with differing perspectives. In order to overcome this difficulty, we have divided each of the main chapters into two parts: the main one, in which we hope the language is understandable by all, contains as few mathematical technicalities as possible yet still states the results in a rigorous way. Then, as needed, a long appendix gives the details of the proofs.

The reader should note that some of the cited original articles underlying this monograph may treat the same problem in two distinct publications: a more physically oriented treatment appearing in a physics or mechanics journal as well as a corresponding "heavy" mathematical treatment presented in a mathematics journal. That is clearly due to the idiosyncrasies of the two kinds of publications and the need for different presentation styles when addressing the different audiences.

A few remarks will conclude this Preface. First, the authors are fully aware that this book is difficult to read because, owing to the nature of the subject, it assumes the reader's familiarity with several distinct areas of knowledge. The three senior authors hope that the younger generation, more accustomed to interdisciplinary work than their predecessors, will find this work more readily accessible than will their elders. In that regard, the three senior authors are delighted that their younger colleague (RR) had agreed to involve himself so deeply in *all* the aspects of this book, and they hope that this bodes well for its future – especially insofar as its interest and accessibility to the younger generation are concerned.

Second, on the anecdotal side, we recall briefly the genesis of this interdisciplinary collaboration. For a number of years CF and RT had worked independently on the analysis of the Navier–Stokes equations; CF had learned the subject from Jacques-Louis Lions and Giovanni Prodi; he collaborated with Prodi and started to develop a rigorous theory of statistical solutions of those equations. RT learned the subject from Jacques-Louis Lions and Jean Leray, and he also worked on the stochastic solutions of the NSE. Then CF and RT met in the summer of 1970 at a meeting – organized by Giovanni Prodi – in Varenna (Italy), and CF visited RT at Orsay (France) in the fall of 1974. Their collaboration started, addressing such different aspects of the NSE as analysis, statistical solutions, and the long time behavior (dynamical systems point of view). At some point RT suggested that their collaboration would become more interesting if they could join forces with a physicist.

By chance in the spring of 1980, Jacqueline Mossino, a former student of RT, met Yvain Trève (OM's co-worker) at a conference on plasma physics in Tucson (Arizona), and contact was established by letter at a time predating e-mail; eventually they met face-to-face for the first time at a meeting in Dekalb (Illinois) in 1981. At that time, OM was working with Trève on finite–mode number approximations of thermal convection satisfying the first and second laws of thermodynamics and discovered that the qualitative nature of the numerical results depended critically on the number of modes retained (Trève and Manley [1982]). As they started to interact, CF, OM, and RT realized immediately the extent of the common ground between the two communities and perspectives that they represent. That realization was the original stimulus for much of the research reported in this volume. More specifically, the direction of that research was set by the recognition that a simple, physically based argument (conservation of energy and momentum in thermal convection; Trève and Manley [1981]) yielded a result – a bound on the sufficient number of degrees of freedom for this fluid flow – that is essentially equivalent to an elaborate mathematical exercise in Sobolev spaces (Foias, Manley, Temam, and Trève [1983b]). This collaboration has extended through the rest of the 1980s, the 1990s, and beyond.

The youngest author was a graduate student at Indiana University from 1992 to 1996, and he had many opportunities to be exposed to this research through courses, informal discussions, and seminar lectures. He enthusiastically agreed to participate in this book, which has been in process for a number of years, and eventually started to collaborate on more recent works. As indicated earlier, the three senior authors are delighted that RR has joined them in this task, and they see it as a good omen for a successful transmittal of these results to the next generation.

Beside the prolonged and extended efforts of the four authors, this book has benefitted extensively from the input and influence of many others by occasional collaborations, discussions, and other forms of interaction. It is not possible to name them all, but we want to thank them for their constructive influence on us. Also, we would like to extend our deepest thanks to those who have co-authored relevant publications with one or more of the authors of this book, works that are partially or fully reported in this monograph: Hari Bercovici, Peter Constantin (with whom three of us had an extended collaboration), Arnaud Debussche, Jean-Michel Ghidaglia, David Gottlieb, Martine Marion, Jean-Claude Saut, George Sell, Denis Serre, Edriss Titi, and Yvain Trève.

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