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The Subgroup Structure of the Finite Classical Groups

Peter Kleidman,
Princeton University

Martin Liebeck
Imperial College, London

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Preface

Following the classification of the finite simple groups, completed in 1980, one of the major areas of research in group theory today is the investigation of the subgroups of the finite simple groups, and in particular, the determination of their maximal subgroups. According to the classification theorem, the finite simple groups fall into four classes:

the alternating groups A_n $(n \geq 5)$;

the finite classical groups — that is, the linear, symplectic, unitary and orthogonal groups on finite vector spaces;

the exceptional groups of Lie type;

the 26 sporadic groups.

In this book we concentrate on the classical groups, which we describe in detail in Chapter 2. Our work takes as its starting point the fundamental results of M. Aschbacher in [As$_1$]. Let G be a finite classical group. In [As$_1$], Aschbacher introduces a large collection $\mathcal{C}(G)$ of natural, geometrically defined subgroups of G, and shows that almost every subgroup of G is contained in a member of $\mathcal{C}(G)$ (the precise result is stated in Chapter 1). Thus the collection $\mathcal{C}(G)$ of subgroups plays a central role in the theory of classical groups. This book is intended to be a definitive investigation of the collection $\mathcal{C}(G)$. In it, we solve the three main problems concerning these subgroups — namely, we determine

(I) the group-theoretic structure of each member of $\mathcal{C}(G)$,

(II) the conjugacy among the members of $\mathcal{C}(G)$,

(III) precisely which members of $\mathcal{C}(G)$ are maximal in G and which are not — and, for non-maximal members H of $\mathcal{C}(G)$, we determine the maximal overgroups of H in G.

Some of these results have been obtained by Aschbacher and others, but many of them are new.

The layout of the book is as follows. In Chapter 1 we present a more detailed motivation and setting for the work in this book. In particular, we describe Aschbacher's theorem, and also survey recent developments in the study of the subgroup structure of the other (non-classical) simple groups.

Chapter 2 contains an introduction to the classical groups. This includes a detailed description of forms and standard bases for classical geometries, and of generators, automorphisms, simplicity and exceptional isomorphisms of classical groups. In §2.10 there is also a compendium of results concerning the representation of a classical group on its associated geometry.

In Chapter 3 we state the main results of the book. These are given in the form of nine tables (Tables 3.5.A-I). The chapter contains explanations of how to use the tables to read off the solutions to problems (I), (II) and (III) above.

Chapter 4 contains the proof of all the results concerning the structure and conjugacy of the subgroups in $\mathcal{C}(G)$. The subgroups in $\mathcal{C}(G)$ are divided into eight families $\mathcal{C}_i(G)$ $(1 \leq i \leq 8)$, and the material in the chapter is accordingly presented in eight sections.

Each section begins with the definition and explicit description of the subgroups in $\mathcal{C}_i(G)$.

The remainder of the book is devoted to the solution of problem (III) — that is, to finding the maximal overgroups of subgroups in $\mathcal{C}(G)$. We handle this problem in two parts: (a) finding overgroups which themselves lie in $\mathcal{C}(G)$ and (b) finding the other overgroups. The overgroups occurring in (a) are constructed in Chapter 6, and in Chapter 7 we prove that these are the only such examples. As for (b), it follows from Aschbacher's theorem (described in Chapter 1) that it suffices to find just those overgroups which are almost simple and absolutely irreducible on the vector space associated with the classical group G. We find all such overgroups in Chapter 8. This chapter is the only one in the book in which the classification theorem is actually used. To employ the classification we require a good deal of detailed information about the simple groups, particularly concerning their permutation actions and their representations. This information is gathered in Chapter 5. One feature of the examples in Chapter 8 is the occurrence of spin modules for symplectic and orthogonal groups in characteristic 2. Some detailed calculations with these modules are required, and the groundwork for these is laid in §5.4 in Chapter 5.

We have aimed to be comprehensible to a first year graduate student with a background in algebra. We have also tried to present our material in such a way as to be useful to any research worker using simple groups, even if he or she is not especially interested in subgroup structure. In particular, we believe that Chapter 2 will serve as a useful introduction to the basic properties of classical groups, and Chapter 5 as a guide to the methods available when solving problems using the classification theorem.

Standard notation and terminology

Group-theoretic notation

Let G and H be finite groups, p a prime and n an integer.

$H \preceq G$	H is isomorphic to a subgroup of G
$H.G$	extension of H by G
$H{:}G$	split extension of H by G
$H \circ G$	central product of H and G
$\frac{1}{n}G$	normal subgroup of index n in G
G'	derived group of G
G^∞	last term of the derived series of G
$m_p(G)$	p-rank of G
$Syl_p(G)$	set of Sylow p-subgroups of G
$\|G\|_p$	power of p dividing $\|G\|$
$O_p(G)$	largest normal p-subgroup of G
$O^p(G)$	subgroup of G generated by all its p'-elements
$\|g\|$	order of element $g \in G$
$Z(G)$	centre of G
$\mathrm{soc}(G)$	subgroup of G generated by its minimal normal subgroups
$\mathbf{Z}_n$ or just n	cyclic group of order n
$\mathbf{Z}_p^n$ or just p^n	elementary abelian group of order p^n
$[n]$	arbitrary soluble group of order n
quasisimple group	perfect group G such that $G/Z(G)$ is non-abelian simple
component of G	quasisimple subnormal subgroup of G

Other notation

$\mathbf{F}_q$	field of q elements
$\mathbf{F}^*$	non-zero elements in the field $\mathbf{F}$
$(\mathbf{F}^*)^n$	n^{th} powers in $\mathbf{F}^*$
$T^{\mathbf{F}}_{\mathbf{F}_o}, N^{\mathbf{F}}_{\mathbf{F}_o}$	trace and norm maps $\mathbf{F} \mapsto \mathbf{F}_o$ where $\mathbf{F}_o$ is a subfield of $\mathbf{F}$
$\mathrm{Gal}(\mathbf{F}{:}\mathbf{F}_o)$	Galois group of $\mathbf{F}$ over $\mathbf{F}_o$
A^{t}	transpose of the matrix A
$(a_1, \ldots, a_n)$	highest common factor of the integers $a_1, \ldots, a_n$
$[a, b]$	lowest common multiple of the integers a, b
$[x]$, $\lfloor x \rfloor$	greatest integer $\leq x$
$\lceil x \rceil$	smallest integer $\geq x$
$x - \epsilon$	used for $x - \epsilon 1$ where $\epsilon = +$ or $-$

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