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Chapter 1
MOTIVATION AND SETTING FOR THE RESULTS

§1.1 Introduction

In this chapter we provide the motivation and setting for this book by surveying recent developments in the study of the subgroup structure of finite groups. We also provide, in §1.2 below, a rough statement of the results which will be proved in the later chapters (see Theorem 1.2.3).

By the Jordan-Hölder Theorem, every finite group is 'built' out of simple groups, and so questions about arbitrary finite groups often reduce to questions about simple groups or *almost simple* groups — here, a group G is said to be *almost simple* if

$$G_o \trianglelefteq G \leq Aut(G_o) \qquad (1.1.1)$$

for some non-abelian simple group G_o. Much of the information about a group can be gleaned from a study of its subgroups. For these reasons it is important to study the subgroup structure of the almost simple groups, and in particular their *maximal* subgroups. This subject has a rich history, dating back as far as Galois' famous letter to Chevalier, written on the eve of his ill-fated duel in 1832 [Ga_1]. In recent years there have been far-reaching developments in the area, which we now describe. Throughout this chapter, G_o denotes a finite non-abelian simple group and G a group satisfying (1.1.1). The maximal subgroups of G which contain G_o are not of interest to us, for they correspond merely to the subgroups of $Out(G_o) = Aut(G_o)/G_o$, which is a soluble group with a transparent structure. Thus throughout Chapter 1, the subgroups of G discussed are assumed not to contain G_o.

The non-abelian finite simple groups have been competely classified (for a discussion of this classification see [Go], for example), and they fall into four families:

(1) the alternating groups A_n ($n \geq 5$);
(2) the finite classical groups — that is, the linear, symplectic, unitary and orthogonal groups on finite vector spaces;
(3) the exceptional groups of Lie type;
(4) the 26 sporadic groups.

The families under (3) and (4) can be regarded as 'groups of bounded rank', and the study of their subgroups requires methods significantly different from those used for the groups in (1) and (2). We describe some recent work on the subgroups of groups in (3) and (4) briefly in §1.3 below. For the families (1) and (2) of 'unbounded rank', there are powerful *subgroup structure theorems* available, due to O'Nan and Scott (see [Sc, Appendix] and [A-S, Appendix]) for the alternating groups, and to Aschbacher [As_1] for the classical groups. In these structure theorems, a 'natural' collection of subgroups of G is defined (where G satisfies (1.1.1) and G_o is alternating or classical, and it is proved that for any subgroup H of G not containing G_o,

> *either* H is contained in a member of the natural collection,
>
> *or* $H \in \mathcal{S}$, where $\mathcal{S}$ is a set of almost simple subgroups of G satisfying certain 'irreducibility' conditions.

We call the natural collection $\mathcal{A}(G)$ in the alternating case and $\mathcal{C}(G)$ in the classical case, and these along with $\mathcal{S}$ are described in §§1.2-1.3.

As a consequence of these subgroup structure theorems, every maximal subgroup of G lies in $\mathcal{A}(G), \mathcal{C}(G)$ or $\mathcal{S}$. Thus to obtain a classification of the maximal subgroups of G, one seeks to determine when the subgroups in $\mathcal{A}(G), \mathcal{C}(G)$ and $\mathcal{S}$ are in fact maximal in G. This programme is complete in the alternating case [L-P-S$_1$], and those results are outlined in §1.3. It is the purpose of this book to contribute to this programme for the classical groups. In a strong sense, a large portion of the maximal subgroups of a classical group G lies in $\mathcal{C}(G)$, and we shall solve three main problems concerning $\mathcal{C}(G)$ — namely, we determine precisely which members of $\mathcal{C}(G)$ are maximal in G and which are not, the exact group-theoretic structure of each member of $\mathcal{C}(G)$, and the conjugacy amongst the groups in $\mathcal{C}(G)$. We now outline these results and some consequences. The precise results are stated in Chapter 3.

§1.2 The classical groups

The reader unfamiliar with the classical groups is referred to Chapter 2, where notation is established and definitions and basic properties of these groups are given. Let G_o be one of the following classical simple groups:

$$\begin{gathered} PSL_n(q),\ PSU_n(q),\ PSp_n(q)\ (n \text{ even}), \\ P\Omega_n^{\pm}(q)\ (n \text{ even}),\ \Omega_n(q)\ (nq \text{ odd}). \end{gathered} \tag{1.2.1}$$

Write V for the natural n-dimensional vector space over the finite field $\mathbf{F}$ associated with G_o (so that $\mathbf{F} = \mathbf{F}_q$ for the linear, symplectic and orthogonal groups and $\mathbf{F} = \mathbf{F}_{q^2}$ for the unitary groups), and write $q = p^f$, where p is prime. Also let Γ be the full semilinear classical group corresponding to G_o. For example, if $G_o = PSL_n(q)$, then $\Gamma = \Gamma L_n(q)$. The group Γ contains the scalars as a normal subgroup, and writing $\overline{}$ for reduction modulo scalars we see that $G_o \trianglelefteq \overline{\Gamma} \leq Aut(G_o)$, and $\overline{\Gamma} = Aut(G_o)$ except in the cases which we mention after Theorem 1.2.1, below.

The subgroup structure theorem for the classical groups is due to Aschbacher [As$_1$]. In [As$_1$], eight collections $\mathcal{C}_i(\Gamma)$ $(1 \leq i \leq 8)$ of natural subgroups of Γ are defined. The precise definitions and structures of the groups in $\mathcal{C}_i(\Gamma)$ are given in Chapter 4, so here we content ourselves with a 'rough' description in the left hand side of Table 1.2.A. In the right hand side we illustrate by giving the approximate structures of the subgroups $H \cap GL_n(q)$, where $H \in \mathcal{C}_i(\Gamma)$ and $G_o = PSL_n(q)$. Some of the notation used in the table will be discussed in Chapters 2 and 4.

Table 1.2.A

$\mathcal{C}_i$	rough description	rough structure in $GL_n(q)$
$\mathcal{C}_1$	stabilizers of totally singular or non-singular subspaces	maximal parabolic
$\mathcal{C}_2$	stabilizers of decompositions $V = \bigoplus_{i=1}^{t} V_i,\ \dim(V_i) = a$	$GL_a(q) \wr S_t,\ n = at$
$\mathcal{C}_3$	stabilizers of extension fields of $\mathbf{F}_q$ of prime index b	$GL_a(q^b).b,\ n = ab,\ b$ prime
$\mathcal{C}_4$	stabilizers of tensor product decompositions $V = V_1 \otimes V_2$	$GL_a(q) \circ GL_b(q),\ n = ab$
$\mathcal{C}_5$	stabilizers of subfields of $\mathbf{F}_q$ of prime index b	$GL_n(q_o),\ q = q_o^b,\ b$ prime
$\mathcal{C}_6$	normalizers of symplectic-type r-groups (r prime) in absolutely irreducible representations	$(\mathbf{Z}_{q-1} \circ r^{1+2a}).Sp_{2a}(r),\ n = r^a$
$\mathcal{C}_7$	stabilizers of decompositions $V = \bigotimes_{i=1}^{t} V_i,\ \dim(V_i) = a$	$\overbrace{(GL_a(q) \circ \cdots \circ GL_a(q))}^{t}.S_t,\ n = a^t$
$\mathcal{C}_8$	classical subgroups	$Sp_n(q),\ n$ even $O_n^\epsilon(q),\ q$ odd $GU_n(q^{1/2}),\ q$ a square

For any subgroup Y of Γ, define $\mathcal{C}_i(Y) = \{C \cap Y \mid C \in \mathcal{C}_i(\Gamma)\}$ for $1 \le i \le 8$, and let

$$\mathcal{C}(Y) = \bigcup_{i=1}^{8} \mathcal{C}_i(Y).$$

Further, put $\mathcal{C}_i(\overline{Y}) = \overline{\mathcal{C}_i(Y)}$ and $\mathcal{C}(\overline{Y}) = \overline{\mathcal{C}(Y)}$. Thus the right-hand column of Table 1.2.A gives the approximate structure of members of $\mathcal{C}_i(GL_n(q))$ when $\Gamma = \Gamma L_n(q)$.

Next we define the class $\mathcal{S}$ of subgroups of G, where G is as in (1.1.1) and G_o as in (1.2.1).

Definition of $\mathcal{S}$. The subgroup H of G lies in $\mathcal{S}$ if and only if the following hold.

(a) The socle S of H is a non-abelian simple group (i.e., H is almost simple).

(b) If L is the full covering group of S, and if $\rho : L \to GL(V)$ is a representation of L such that $\overline{\rho(L)} = S$, then ρ is absolutely irreducible.

(c) $\rho(L)$ cannot be realized over a proper subfield of $\mathbf{F}$.

(d) If $\rho(L)$ fixes a non-degenerate quadratic form on V, then $G_o = P\Omega_n^\epsilon(q)$.

(e) If $\rho(L)$ fixes a non-degenerate symplectic form on V, but no non-degenerate quadratic form, then $G_o = PSp_n(q)$.

(f) If $\rho(L)$ fixes a non-degenerate unitary form on V, then $G_o = PSU_n(q)$.

(g) If $\rho(L)$ does not satisfy the conditions in (d), (e) or (f), then $G_o = PSL_n(q)$.

Remark. Condition (b) in the definition of $\mathcal{S}$ is imposed to ensure that a group H in $\mathcal{S}$ is contained in no member of $\mathcal{C}_3(G)$. Similarly, (c)-(g) are imposed to ensure that H is contained in no member of $\mathcal{C}_5(G) \cup \mathcal{C}_8(G)$. In item (c), the phrase 'realized over a proper subfield' is defined in §2.10 (just before Lemma 2.10.7). As for (d)-(f), we refer the reader to §2.1. For a definition of 'covering group' in (b), see [As$_8$, §33].

We are now in a position to state the main subgroup structure theorem of [As$_1$].

Theorem 1.2.1 (Aschbacher [As$_1$]). *Let G be a group such that $G_o \trianglelefteq G \leq \overline{\Gamma}$, with G_o and Γ as in (1.2.1) above, and let H be a subgroup of G not containing G_o. Then either H is contained in a member of $\mathcal{C}(G)$ or $H \in \mathcal{S}$.*

Theorem 1.2.1 covers almost all groups whose socle is a classical simple group. However, there are three cases in which $\overline{\Gamma} \neq Aut(G_o)$, as follows.

(i) $G_o = PSL_n(q)$ and $n \geq 3$. Here G_o has an 'inverse-transpose' automorphism, called ι say, and $Aut(G_o) = \overline{\Gamma}\langle\iota\rangle$ (see §2.2). In [As$_1$, §13], an extra family $\mathcal{C}_1'(G)$ of subgroups of G is introduced when $G \not\leq \overline{\Gamma}$, and a version of Theorem 1.2.1 is proved using $\mathcal{C}_1'(G)$. As a convenience, we redefine $\mathcal{C}_1(G)$ so as to include $\mathcal{C}_1'(G)$.

(ii) $G_o = Sp_4(q)$ with q even. Here G_o admits a graph automorphism, which is discussed in [As$_1$ (§14),Ca$_1$ (Ch.12),Fl$_2$] and $|Aut(G_o) : \overline{\Gamma}| = 2$. When $G \not\leq \overline{\Gamma}$, the definition of $\mathcal{C}(G)$ is suitably modified in [As$_1$, §14] in such a way that a version of Theorem 1.2.1 still holds. This was also done in [Wi$_6$].

(iii) $G_o = P\Omega_8^+(q)$. Here G_o admits a 'triality' automorphism as described in [Ca$_1$ (Ch.12)] and $|Aut(G_o) : \overline{\Gamma}| = 3$. If $G \not\leq \overline{\Gamma}$, then no conclusive result is obtained in [As$_1$] — however [Kl$_1$] contains a full determination of the maximal subgroups of such groups G.

By Theorem 1.2.1, to obtain a classification of the maximal subgroups of the classical groups, one must consider the maximality of subgroups in $\mathcal{C}(G) \cup \mathcal{S}$. Cases where $n = \dim_{\mathbf{F}}(V)$ is small have been studied for a long time, dating as far back as the famous letter of Galois to Chevalier in 1832, which contains significant observations on the subgroups of $PSL_2(p)$. Many of the classical groups with $n \leq 7$ were dealt with prior to the classification of finite simple groups (see [K-L] for references). Using the classification and Theorem 1.2.1, the following result has now been proved.

Theorem 1.2.2 [Kl$_1$,Kl$_2$]. *Let G_o be as in (1.2.1) above with $n \leq 12$, and let $G_o \trianglelefteq G \leq Aut(G_o)$. Then the maximal subgroups of G are known.*

We now come to the main result of this book, dealing with the maximality of the subgroups in $\mathcal{C}(G)$ in general. In view of Theorem 1.2.2, we can restrict our attention to the case where $n \geq 13$. Put in relatively loose terms, the main result can be stated in the following way.

Main Theorem 1.2.3. *Let G_o be as in* (1.2.1) *and let $G_o \trianglelefteq G \leq Aut(G_o)$. Then*

(A) *the group-theoretic structure of each $H \in \mathcal{C}(G)$ is known;*

(B) *the conjugacy amongst the members of $\mathcal{C}(G)$ is known;*

(C) *for $H \in \mathcal{C}(G)$ all overgroups of H which lie in $\mathcal{C}(G) \cup \mathcal{S}$ are known* (*for* $n \geq 13$).

The precise version of Theorem 1.2.3 is stated as the Main Theorem in §3.1 and Tables 3.5.A-I. Sections 3.2-3.4 describe how these tables are to be read, and how one can obtain from them all the information advertised in Theorem 1.2.3, above. Parts (A) and (B) of Theorem 1.2.3 actually hold for all n, and not just for $n \geq 13$. Several portions of (A) and (B) are addressed in [As_1, Theorems A,B], but a number of significant questions are left unresolved there. As for (C), there is an extensive literature concerning various parts of the problem: for subgroups $H \in \mathcal{C}_1(G) \cup \mathcal{C}_2(G)$, the result has been obtained by geometric methods in [Dy_3,Ke_1,Ke_2] and in [Ki_1,Ki_2,Ki_3,Ki_4,Ki_5]. The maximality of subgroups in $\mathcal{C}_5(G) \cup \mathcal{C}_8(G)$ has been established in certain cases in [B-G-L,Dy_1,Dy_2,Ki_6,Ki_7,Mc,Po]. Some subgroups in $\mathcal{C}_3(G)$, and some in $\mathcal{C}_4(G)$, are treated in [Dy_4,Li_1,Li_2]. For the remaining cases — $\mathcal{C}_6(G), \mathcal{C}_7(G)$ and parts of $\mathcal{C}_3(G), \mathcal{C}_4(G), \mathcal{C}_5(G)$ — our result is new. We shall include complete proofs for all the groups in $\mathcal{C}(G)$, since our proofs are fairly uniform, and there is not much advantage to be gained by omitting various previously known special cases. However, it should be noted that we make use of the classification of finite simple groups in our proofs, whereas none of the above references does. (Note, however, that we use the classification only in Chapter 8 where we determine the overgroups in $\mathcal{S}$ of subgroups in $\mathcal{C}$.) In order to prove Theorem 1.2.3, we require a wealth of information about the structure and representations of the finite simple groups. We summarize this information in Chapter 5.

Consequences of Theorem 1.2.3

When studying the maximal subgroups of G (which do not contain G_o), one often distinguishes two general cases: the local case and the non-local case. More specifically, let H be such a maximal subgroup of G and set $H_o = H \cap G_o$. With a slight abuse of terminology, we call H *local* if $O_r(H_o) \neq 1$ for some prime r, and *non-local* otherwise. Since by Theorem 1.2.1 any local maximal subgroup of G obviously lies in $\mathcal{C}(G)$, we have the first main consequence of Theorem 1.2.3.

Corollary 1.2.4. *The local maximal subgroups of the finite classical groups are known.*

The precise list of local maximal subgroups can be read off from the Propositions in Chapter 4. This result, together with various results concerning the local maximal subgroups of the alternating, exceptional and sporadic groups discussed in §1.3, yields the following

Theorem 1.2.5. *The local maximal subgroups of all the finite simple groups are known, apart from the 2-locals of the Monster and the Baby Monster.*

Second, as mentioned in §1.1, Theorem 1.2.3 reduces a classification of the maximal subgroups of the classical groups to the problem of determining when groups in $\mathcal{S}$ are maximal in G. Work on this problem is in an interesting state at present, and we briefly describe some recent results. Let $H \in \mathcal{S}$, so that $S \trianglelefteq H \leq Aut(S)$ for some non-abelian simple group S, with S absolutely irreducible on V. The goal is to classify all such groups H which fail to be maximal in $HG_o = G$. Now if H is non-maximal in G, then $H < K < G$ for some maximal subgroup K of G. According to Theorem 1.2.1, we have $K \in \mathcal{C}(G) \cup \mathcal{S}$, and it is clear from the definitions that H is not contained in a member of $(\mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_8)(G)$ (see the remark after the definition of $\mathcal{S}$). Thus two situations can arise:

(i) $K \in (\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_7)(G)$;

(ii) $K \in \mathcal{S}$.

If S is not of Lie type in characteristic p (here p is the defining characteristic of G_o), then first steps have been achieved in [Se3]. However, for the most part, relatively little is known about the possibilities for H and K; for example classifying all instances of $H < K \in \mathcal{C}_4(G)$ is presently intractable, for it amounts to classifying all absolutely irreducible representations of the quasisimple groups which are tensor decomposable. So let us suppose that S is of Lie type in characteristic p. The p-modular representation theory of such groups S is sufficiently advanced to gain good control over situation (i), and so let us focus our attention on situation (ii). We may write $T \trianglelefteq K \leq Aut(T)$ with T non-abelian simple, and we have

$$S < T < G_o.$$

Consequently the aim here is to classify all such triples (S, T, G_o). The main theorem of [L-S-S] determines all but finitely many of the triples where S is of Lie type in characteristic p while T is not. In the case where both S and T are both of Lie type in characteristic p, significant advances have been achieved by Seitz and Testerman [Se1,Se2,Te].

§1.3 The alternating, sporadic and exceptional groups

We complete this survey by outlining the present state of knowledge on maximal subgroups of the alternating groups, the sporadic groups and the exceptional groups of Lie type.

We first consider the alternating groups A_n. As mentioned in §1.1, there is a subgroup structure theorem, due to O'Nan and Scott (see [Sc, Appendix]), which is analogous to Aschbacher's theorem for the classical groups. Let A_n and S_n act naturally on $I = \{1, \ldots, n\}$. As for the classical groups, we now describe five collections $\mathcal{A}_i$ $(1 \leq i \leq 5)$ of natural subgoups of S_n in Table 1.3.A, below.

Table 1.3.A

$\mathcal{A}_i$	rough description	structure in S_n	comments
$\mathcal{A}_1$	stabilizers of subsets of I	$S_a \times S_b$	$n = a + b,\ a \neq b$
$\mathcal{A}_2$	stabilizers of partitions of I into subsets of equal size	$S_a \wr S_b$	$n = ab,\ a \geq 2,\ b \geq 2$
$\mathcal{A}_3$	stabilizers of affine structures on I	$AGL_d(p)$	$n = p^d$, p prime
$\mathcal{A}_4$	stabilizers of cartesian product structures on I	$S_a \wr S_b$	$n = a^b,\ a \geq 5,\ b \geq 2$
$\mathcal{A}_5$	the normalizer of T^k acting on the cosets of a diagonal subgroup, where T is a non-abelian simple group	$T^k.(Out(T) \times S_k)$	$n = \vert T\vert^{k-1},\ k \geq 2$

For a more detailed discussion of the families $\mathcal{A}_i$ consult [L-P-S$_2$].

For groups G satisfying $A_n \leq G \leq S_n$, define $\mathcal{A}_i(G) = \{A \cap G \mid A \in \mathcal{A}_i\}$ for $1 \leq i \leq 5$, and let

$$\mathcal{A}(G) = \bigcup_{i=1}^{5} \mathcal{A}_i(G).$$

The class $\mathcal{S}$ of subgroups of G consists of those almost simple groups acting primitively on I. With this notation, the subgroup structure theorem for the alternating groups may be stated as follows.

Theorem 1.3.1 (O'Nan-Scott [Sc]). *Let G be A_n or S_n and let H be a subgroup of G not containing A_n. Then either H is contained in a member of $\mathcal{A}(G)$ or $H \in \mathcal{S}$.*

Thus we may obtain a classification of the maximal subgroups of the alternating and symmetric groups by determining precisely when the subgroups H in $\mathcal{A}(G) \cup \mathcal{S}$ are maximal in A_nH. This problem is now completely solved:

Theorem 1.3.2 [L-P-S$_1$]. *Let G be A_n or S_n and let $H \in \mathcal{A}(G) \cup \mathcal{S}$. Then either H is maximal in A_nH, or $H < K < A_nH$, where (H, K, n) is given in an explicit list of triples.*

As far as the sporadic groups are concerned, the maximal subgroups of all but three of them are completely determined (the three exceptions are Fi_{24}', B and M). Most of the lists of maximal subgroups can be found in [At] and a recent survey of the subject appears in [Wi$_1$].

For the exceptional groups of Lie type, we list in Table 1.3.B those groups G_o for which the maximal subgroups of G are known (where as usual $G_o \trianglelefteq G \leq Aut(G_o)$).

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Table 1.3.B

G_o	references
${}^2B_2(q)$	[Su_1]
${}^2G_2(q)$	[L-N,Kl_3]
$G_2(q)$	[As_2,Co_2,Kl_3,Mi]
${}^3D_4(q)$	[Kl_4]
${}^2F_4(2)'$	[Tc,Wi_3]
${}^2F_4(q)'$	[Ma]
$F_4(2)$	[N-W]
${}^2E_6(2)$	[No]
$E_6(2)$	[K-W_3]

In addition to the references appearing in Table 1.3.B, the maximal subgroups of $G_2(4)$ were obtained independently in [Bu,P-T,Wi_2], and those of ${}^3D_4(2)$ by Wilson, whose result appears in [At]. Recently Aschbacher [As_3,As_4,As_5,As_6,As_7] has obtained a wealth of information concerning the maximal subgroups of G when G_o is the simple group $E_6(q)$ and G does not induce a graph automorphism on G_o. Here, almost all the maximal subgroups have been determined, with the possibility of a few 'small' exceptions left open. These results have been extended to the case where G does induce a graph automorphism in [Kl_5], and this has led to a subgroup theorem for groups with socle ${}^2E_6(q)$.

As we stated in Theorem 1.2.5, it is a fact that all local maximal subgroups of the exceptional groups are known. Of course, each reference already mentioned in this section contributes to the proof of this fact. However, the complete list of the local maximal subgroups has been obtained in [C-L-S-S].

Finally, we remark that the recent paper [L-Se], following work of Borovik [Bor], reduces the study of maximal subgroups of the exceptional groups of Lie type to the study of almost simple maximal subgroups, and even handles these subgroups in some cases for large characteristics.

Chapter 2
BASIC PROPERTIES OF THE CLASSICAL GROUPS

§2.1 Introduction

In this chapter we construct the classical groups and describe their fundamental properties. We begin by introducing some basic definitions, terminology and notation.

Throughout, V will denote a vector space of finite dimension n over the field $\mathbf{F}$, where $\mathbf{F}$ is either a finite field or an algebraically closed field of characteristic p. We write $GL(V,\mathbf{F})$ for the *general linear group* of V over $\mathbf{F}$, which is the group of all non-singular $\mathbf{F}$-linear transformations of V. Also $SL(V,\mathbf{F})$ denotes the *special linear group* of V over $\mathbf{F}$, the group of elements in $GL(V,\mathbf{F})$ with determinant 1. Loosely speaking, the classical groups are the stabilizers in $GL(V,\mathbf{F})$ and $SL(V,\mathbf{F})$ of suitable forms on V, such as non-degenerate symmetric, skew-symmetric or sesquilinear forms. In the following sections we will provide a technical account of the classical groups. Other references for the classical groups include [Ar,As$_8$,Di$_1$,Ka$_2$].

We usually reserve the letter β for a basis of V. If $\beta = \{v_1,\ldots,v_n\}$ is such a basis, then each element of $GL(V,\mathbf{F})$ is determined by its action on β. If $g \in GL(V,\mathbf{F})$, then g_β will denote the $n \times n$ matrix which satisfies $v_i g = \sum_{j=1}^n (g_\beta)_{ij} v_j$ and if $X \subseteq GL(V,\mathbf{F})$ then $X_\beta = \{g_\beta \mid g \in X\}$. For $\lambda_i \in \mathbf{F}^*$ $(i = 1,\ldots,n)$, we denote by $\mathrm{diag}_\beta(\lambda_1,\ldots,\lambda_n)$ the *diagonal* linear transformation which satisfies $v_i \mathrm{diag}_\beta(\lambda_1,\ldots,\lambda_n) = \lambda_i v_i$. If the λ_i are all equal to λ say, then $\mathrm{diag}_\beta(\lambda,\ldots,\lambda)$ is called a *scalar linear transformation,* or simply a *scalar*. The centre of $GL(V,\mathbf{F})$ is the group of all non-zero scalars, which is isomorphic to $\mathbf{F}^*$. So with a slight abuse of notation we write $\mathbf{F}^* \leq GL(V,\mathbf{F})$, and the scalar $\mathrm{diag}_\beta(\lambda,\ldots,\lambda)$ is denoted simply by λ. We also write $PGL(V,\mathbf{F})$ for the *projective general linear group* $GL(V,\mathbf{F})/\mathbf{F}^*$. And if X is any subgroup of $GL(V,\mathbf{F})$, then we write PX for the corresponding projective group $X/X \cap \mathbf{F}^*$. Thus for example $PSL(V,\mathbf{F})$ is the *projective special linear group*. Along with P, the symbol $\overline{}$ will also denote reduction modulo scalars. Thus $\overline{GL(V,\mathbf{F})} = PGL(V,\mathbf{F})$, and if $g \in GL(V,\mathbf{F})$ then $\overline{g}$ denotes the image of g in $PGL(V,\mathbf{F})$.

If V' is another vector space over $\mathbf{F}$, then $\mathrm{Hom}_\mathbf{F}(V,V')$ denotes the set of $\mathbf{F}$-linear maps from V to V'. Moreover we put $\mathrm{End}_\mathbf{F}(V) = \mathrm{Hom}_\mathbf{F}(V,V)$. For any subset $X \subseteq \mathrm{End}_\mathbf{F}(V)$, we write $\mathrm{End}_{\mathbf{F}X}(V)$ for the subset of $\mathrm{End}_\mathbf{F}(V)$ commuting with all elements of X.

Semilinear transformations

A map g from V to V is called an $\mathbf{F}$-*semilinear transformation of* V if there is a field automorphism $\sigma(g) \in Aut(\mathbf{F})$ such that for all $v,w \in V$ and $\lambda \in \mathbf{F}$,

$$(v+w)g = vg + wg \quad \text{and} \quad (\lambda v)g = \lambda^{\sigma(g)}(vg). \tag{2.1.1}$$

If g is an $\mathbf{F}$-semilinear transformation, then g is *non-singular* if $\{v \in V \mid vg = 0\} = \{0\}$. Now define $\Gamma L(V, \mathbf{F})$ as the set of all non-singular $\mathbf{F}$-semilinear transformations of V. It is easy to verify that if $g, h \in \Gamma L(V, \mathbf{F})$, then their composition gh also lies in $\Gamma L(V, \mathbf{F})$ and $\sigma(gh) = \sigma(g)\sigma(h)$. Consequently $\Gamma L(V, \mathbf{F})$ forms a group, called the *general semilinear group* of V over $\mathbf{F}$, and the map σ from $\Gamma L(V, \mathbf{F})$ to $Aut(\mathbf{F})$ is a surjective homomorphism with kernel $GL(V, \mathbf{F})$.

Obviously $\mathbf{F}^* \trianglelefteq \Gamma L(V, \mathbf{F})$ and upon factoring out the scalars we obtain the *projective general semilinear group* $P\Gamma L(V, \mathbf{F}) = \Gamma L(V, \mathbf{F})/\mathbf{F}^*$. As before we may define PX and $\overline{X}$ for any subset X of $\Gamma L(V, \mathbf{F})$.

Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V over $\mathbf{F}$ as above. Just as each element of $GL(V, \mathbf{F})$ is determined by its action on β, so is each element $g \in \Gamma L(V, \mathbf{F})$ determined by its action on β along with $\sigma(g)$. If $\alpha \in Aut(\mathbf{F})$ then we define $\phi_\beta(\alpha)$ as the (unique) element of $\Gamma L(V, \mathbf{F})$ which lies in $\sigma^{-1}(\alpha)$ and which fixes each v_i. Thus

$$\left(\sum_{i=1}^{n} \lambda_i v_i\right) \phi_\beta(\alpha) = \sum_{i=1}^{n} \lambda_i^\alpha v_i. \tag{2.1.2}$$

Forms and isometry groups

A map $\mathbf{f} : V \times V \to \mathbf{F}$ is a *left-linear form* if for each $v \in V$, the map $V \to \mathbf{F}$ given by $x \mapsto \mathbf{f}(x, v)$ is a linear map. In other words, for all $x, y, v \in V$ and $\lambda \in \mathbf{F}$, we have

$$\mathbf{f}(x + y, v) = \mathbf{f}(x, v) + \mathbf{f}(y, v) \quad \text{and} \quad \mathbf{f}(\lambda x, v) = \lambda \mathbf{f}(x, v). \tag{2.1.3}$$

There is an analogous definition for *right-linear form*. A map which is both a left-linear and a right-linear form is called a *bilinear form*. For any map $Q : V \to \mathbf{F}$, define $\mathbf{f}_Q : V \times V \to \mathbf{F}$ by

$$\mathbf{f}_Q(v, w) = Q(v + w) - Q(v) - Q(w). \tag{2.1.4}$$

Then Q is called a *quadratic form* provided

$$\begin{gathered} Q(\lambda v) = \lambda^2 Q(v) \text{ for all } v \in V \text{ and } \lambda \in \mathbf{F} \\ \text{and} \\ \mathbf{f}_Q \text{ is a bilinear form.} \end{gathered} \tag{2.1.5}$$

When Q is a quadratic form, $\mathbf{f}_Q$ will be called the *associated bilinear form.*

If $\mathbf{f}$ is any map from $V \times V$ to $\mathbf{F}$ and if $\beta = \{v_1, \ldots, v_n\}$ is any basis of V, then we define $\mathbf{f}_\beta$ as the matrix satisfying

$$(\mathbf{f}_\beta)_{ij} = \mathbf{f}(v_i, v_j).$$

Furthermore, for $v \in V$ we call $\mathbf{f}(v, v)$ the $\mathbf{f}$-*norm of* v. In many circumstances we will omit the symbol $\mathbf{f}$ by writing simply (v, w) instead of $\mathbf{f}(v, w)$, and referring to the norm of v, rather than the $\mathbf{f}$-norm.