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Nonstandard Analysis and its Applications

Edited by

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CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York New Rochelle Melbourne Sydney

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521351096

First published 1988
Re-issued in this digitally printed version 2008

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Nonstandard Analysis and its applications.
(London Mathematical Society student texts ; 10)
Papers presented at a conference held at the University of Hull in 1986.
Includes bibliographical references and index.
1. Mathematical analysis, Nonstandard--Congresses.
I. Cutland, Nigel. II. Series.
QA299.82.I56 1988 519.4 88-16194

ISBN 978-0-521-35109-6 hardback
ISBN 978-0-521-35947-4 paperback

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PREFACE

ex asperis per asteriscos

The methods of Abraham Robinson's Nonstandard (or Infinitesimal) Analysis (NSA) are currently being used across the whole spectrum of mathematics - from 'pure' mathematics through to mathematical physics. This book is designed as an introduction to NSA and to some of its many applications, with the working mathematician or student particularly in mind. It has emerged from a conference with the same title held at the University of Hull in 1986, which had the aim of making NSA more widely known in the mathematical community through a series of introductory lecture courses and lectures on current research. The first part of this book consists of papers based on the introductory lectures given at the conference by Tom Lindstrøm, Ward Henson, Jerry Keisler and Sergio Albeverio. The latter part of the book contains papers that present a sample of recent developments in the more advanced applications of NSA.

Lindstrøm's *An Invitation to Nonstandard Analysis* expounds the foundations of the theory. It is designed to be "a friendly welcome requiring no other background than a smattering of general mathematical culture", offered in the belief that NSA "is of greater interest to the analyst than to the logician". Lindstrøm writes "I have tried to make the subject look the way it would had it been developed by analysts or topologists and not logicians." To this end, his presentation of NSA is somewhat different from others in the literature, in that he builds a nonstandard universe and shows how to practice NSA without any use of logic. Then, in the final chapter of his article, he shows how the language of logic is the natural way to explain and codify in a general way what has been going on in the earlier development.

The choice of topics covered in Lindstrøm's *Invitation* is fairly conventional, and is designed to bring the reader to the point where he can study more specialised nonstandard papers with only an occasional consultation of the literature, and where he can begin to think of making applications in his own field of interest.

One of the most fruitful applications of NSA is in measure theory and probability theory, stemming from the discovery of the Loeb measure construction; this is a simple way to construct a rich class of standard measure spaces from nonstandard spaces, discovered by Peter Loeb in 1975. The article by Keisler discusses applications of Loeb measures to problems in probability theory and the theory of stochastic processes, and explains both how and why it is so successful. Attention is restricted to hyperfinite Loeb spaces, which are particularly easy to work with; it is shown that nothing is lost by working with such spaces, since they have very strong properties (homogeneity and universality) that make them more than adequate for any applications in probability theory.

In functional analysis the construction of nonstandard hulls plays a role similar to that of the Loeb construction in measure theory. Nonstandard hulls are standard topological vector spaces that are constructed in a natural way from nonstandard spaces; they have been used in a variety of ways to solve problems in functional analysis. The article by Henson introduces the nonstandard hull construction for topological vector spaces and operators on them, and is designed to serve as both and introduction and a complement to an earlier survey paper of Henson & Moore[1], so as together to provide a comprehensive discussion of the use of NSA in functional analysis. The earlier survey concentrates on Banach spaces; some recent developments in this area are reported here.

Albeverio's article gives an introduction to the many applications of nonstandard methods in mathematical physics. This field has long been seen as a natural one for such applications, because of the way in which NSA can provide new mathematical models of physical phenomena that are perhaps closer to reality. For example, large finite collections of particles may be more accurately modelled by a hyperfinite set (i.e. a set that is infinite, but finite from the nonstandard point of view, and thus inherits many of the properties of finite sets) than by the continuum. Moreover, the nonstandard framework, with genuine

[1]Henson C.W. & Moore, L.C. Jr. (1983). Nonstandard analysis and the theory of Banach spaces; in *Springer Lecture Notes in Mathematics* **983**, 27-112.

infinitesimals and infinite numbers, often allows heuristic reasoning to be made precise in a way that the standard framework prevents. Albeverio's article surveys the different kinds of nonstandard approach that have been productive in mathematical physics, and discusses some specific examples of the kind of results that have been obtained.

The work presented in Loeb's paper is both an alternative approach to the Loeb measure construction and a generalisation of it; he begins with a nonstandard lattice of functions (which could, for example be the measurable functions on a nonstandard measure space) and shows how to construct from it a space of integrable functions. Some recent work extending this approach to vector valued functions is also discussed; here there is an interesting interplay between the Loeb construction and the nonstandard hull construction.

An important but perhaps relatively less well known field of application of nonstandard methods is that of algebra and its interface with the mathematical theory of computation, which is exemplified in the contribution by Benninghofen and Richter. Following a pattern familiar in other applications, the nonstandard approach is used to construct an 'ideal' object (in this case an extension of a free nilpotent group, given as the nonstandard hull of the original group) that is suitably explicit and tractable for the purposes in hand.

The paper of Diener and Stroyan is designed both to introduce *Internal Set Theory* (IST) - an alternative axiomatisation of nonstandard analysis due to Nelson[2] - and to explain the relationship between this and the superstructure approach expounded by Lindstrøm. A slightly restricted version of IST is shown to be valid in a superstructure, and the principal axioms of IST are shown to be equivalent to useful quantifier manipulation rules. It is hoped that the discussion here of the common ground shared between the two approaches will aid the mutual understanding of those familiar with one or other of the dialects[3] of NSA.

A large body of work on the infinitesimal analysis of differential equations has been done over the past ten years or so. The

[2] Nelson, E. (1977). Internal set theory, *Bull. Amer. Math. Soc.* **83**, 1165-1193.

[3] This term was used by R. Anderson in his review of Lutz & Goze, *Nonstandard Analysis*, Lecture Notes in Mathematics **881**, Springer, 1981, which appeared in *Bull. London Math. Soc.* **15**(1983), 94-5.

article by the Dieners (written in IST) is an example of the elegance and fruitfulness of nonstandard methods in this area, in particular in the study of singular perturbations. These are naturally represented as nontrivial perturbations by an infinitesimal.

Stroyan's article explores the way in which the theory of superinfinitesimals (due to Benninghofen and Richter) can be used to analyse the detailed structure of the monads of certain topologies arising in functional analysis. This analysis is then applied to obtain new and delicate results for these topologies.

The final paper in the volume, by Arkeryd, surveys the results he has obtained over a number of years on the Boltzmann equation. A nonstandard model of space and time provides the framework for new existence results for this famous equation; here is a further example of one of the themes discussed by Albeverio.

I should like to offer my sincere thanks to each of the authors for their contribution to this book, and also for their key part in the Hull conference from which it has emerged. This is also an opportunity to thank Tom Lindstrøm and Ward Henson, my co-organisers of the conference, and David Ross who was a great help too.

It is a pleasure to acknowledge the generous support for the meeting that was received from the British Logic Colloquium, the Logic Trust, the London Mathematical Society and the SERC.

This volume would not have seen the light of day but for the dedicated services of Eileen Freeman, who battled away with the manuscript on our new T^3 wordprocessor; I am most grateful for all her efforts and patience. David Ross helped us to tame T^3, and read through a copy of the final version of the manuscript, as did Marek Capiński: many thanks are due to both.

Finally, my wife Mary has been most supportive and patient when I have been engrossed in this project: I owe her a big thank you too.

Hull, April 1988 Nigel Cutland

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